2 Lecture 2

2.1 Sophistication and Naivety

So far, we have cheated a little bit. If you think back to where we started, we said that the data we had was *choices over menus*, yet when discussing the Gul Pesendorfer model, we have been implicitly discussing what people will choose *from* those menus (i.e. the object that maximizies $u(x) + v(x)$). Yet our data set doesn't say anything about that. This is unfortunate because it means we can't say anything about whether the people are trying to model actually understand their temptation and self control issues: at the moment, we cannot model someone who (for example) doesn't know that they have temptation problems - someone that we might call naive.

In order to address this issue, we want to expand our data set to allow us to observe not only people's preferences over menus, but also the preferences (i.e. choices) that they make from these menus. Thus, as well as observing a preference relation $\succeq$ on 2^X, we also observe a preference relation $\trianglerighteq$ on X. You can think of this as coming from an experiment where we first of all ask people to choose between choice sets, so we observe $\succeq$, then we observe them choosing from these choice sets, and so observe $\trianglerighteq$.

Let's assume that $\trianglerighteq$ is a complete, transitive relation. Then we know that there is a utility function $W : X \to \mathbb{R}$ such that $W(x)$ represents $\trianglerighteq$. Furthermore, if our subject satisfies set betweenness (and some other axioms) we know that there is a $u : X \to \mathbb{R}$ and $v : X \to \mathbb{R}$ such that $\succeq$ is represented by

$$U(A) = \max_{x \in A}(u(x) + v(x)) - \max_{y \in A} v(y)$$

The question is, how do we make these two preferences 'tie together'? In other words, how do we make it so that the choice that the subject anticipates when choosing between menus (i.e. $\max_{x \in A}(u(x) + v(x))$) is the choice that they actually make when choosing from those menus (i.e. $\max_{x \in A} W(x)$).

It turns out that the answer is a very intutive axiom called 'sophistication'. The logic behind the sophistication axiom is as follows: under what circumstances would a decision maker strictly prefer an option x added to the choice set? It has to be that they are actually going to choose that item in the second stage. If not, then adding that option is, at best, not going to make the choice set any better, and might make it worse. This gives us the following axiom:

Axiom 2 (Sophistication) *If* $\{x\} \cup A \succ A$*, then it must be the case that* $x \rhd y$ *for all* $y \in A$

In words, if you stricly prefer a choice set that adds x to A, it better be that you will choose x from the union of the two sets - i.e. that x is preferred to eveything in A. It can be proved (though we will not do so) that, in the Gul-Pesendorfer model, this implies that $W(x) = u(x) + v(x)$ (or rather that $u(x) + v(x)$ represent the same ordering as $W(x)$)

It is also clear that a naive person that ignores their temptation will violate this axiom. Consider a decision maker who, when choosing over choice sets, is naive, in the sense that they ignore the effect of temptation - i.e. they assume temptation is zero for every option (we will represent these preferences by $\bar{U}$). If offered the choice between the menu of burger and salad or burger alone then they will choose the burger and salad, as

$$\begin{aligned} & \bar{U}(\{b,s\}) \\ = & \max_{x \in \{s,b\}} u(x) \\ = & u(s) = 4 \end{aligned}$$

while

$$\begin{aligned} & \bar{U}(\{b\}) \\ = & u(b) \\ = & 1 \end{aligned}$$

yet, assuming that the decision maker actually does have temptation, then they will choose the burger from the set $\{s,b\}$ as

$$\begin{aligned} & u(b) + v(b) \\ = & 5 \\ > & 4 = u(s) + v(s) \end{aligned}$$

Thus, this person will violate sophistication, as $\{s,b\} \succ \{b\}$, yet $b \rhd s$. Such a naive agent would potentially pay to have salad included in their choice set, yet will never choose to consume it. Furthermore, a completely naive agent will never pay for commitment.

2.2 Lack of Self Control and Hyperbolic Discounting

An extreme version of this model is one in which the DM has no self control: They always choose the most tempting object in the second stage. This extreme model - sometimes called the Strolz model (after a paper written by Strolz in 1955) is a special case of the Gul Pesendorfer model which can be written as

$$U(A) = \{\max u(x) | x \in A \text{ and } v(x) \geq v(y) \ \forall \ y \in A\}$$

Thus the DM in the second stage always chooses the most tempting option, only maximizing u if there are two items that are equally tempting. This model is the basis of some heavily used models of temptation in macro, such as hyperbolic discounting models.

One version of this model that has been extremely widely used in the applied literature is the so called $\beta - \delta$ model of hyperbolic discounting. This type of model can be used to explain 'present bias', as described in the introductory lecture. As a reminder, a typical experiment describing present bias looks like the following (asked to a thirsty subject)

Question 1 Would you prefer 2 squirts of juice immediately or 3 squirts of juice in 10 minutes?

Question 2 Would you prefer 2 squirts of juice in 10 minutes or 3 squirts of juice in 20 minutes?

'Present bias' is the phenomena by which people tend to choose the earlier, smaller amount in question 1, and the later, higher amount in question 2.

The standard way to model *intertemporal choices* of this type is to assume that the decision maker has a utility function $u : \mathbb{R} \rightarrow \mathbb{R}$ which describes the 'instantateous utility' of juice in each period. The total utility of a 'juice stream' (i.e. a sequence of juice squirts recived at different times) Is the sum of these utilities, discounted by some factor δ applied each period, so

$$U(c_1, c_2, c_3) = u(c_1) + \delta u(c_2) + \delta^2 u(c_3) \tag{1}$$

where c_1 is juice immediately, c_2 is juice in 10 minutes and so on. δ is the 'discount factor' or the amount by future rewards are considered less important than immediate rewards. Note that the smaller δ the less important the future.

Let's use this model to address question 1 and 2 above. For question 1, the immediate juice is preferred to the later juice if the following condition holds (for convenience I will normalize $u(0)$ to 0)

$$\begin{aligned} & u(2)+\delta u(0)+\delta^2 u(0) \\ \geq \;& u(0)+\delta u(3)+\delta^2 u(0) \\ \Rightarrow \;& \delta \leq \frac{u(2)}{u(3)} \end{aligned}$$

This formula makes sense: the immediate reward is preferred if the discount factor is small enough to offset the higher utility of getting 3 squirts of juice rather than two.

Now let's consider question 2. Now the earlier reward will be preferred if

$$\begin{aligned} & u(0)+\delta u(2)+\delta^2 u(0) \\ \geq \;& u(0)+\delta u(0)+\delta^2 u(3) \\ \Rightarrow \;& \delta \leq \frac{u(2)}{u(3)} \end{aligned}$$

As if by magic, exactly the same condition determines whether the earlier option is preferred in question 2. Thus, the pattern of choice that is commonly observed in experiments cannot be explained by the standard model.

Clearly, this is not *really* magic. In fact it turns out that this is a fundamental property of the type of discounting we assumed - which we call *exponential discounting.* This implies that the preference between an amount x at a date t or an amount y at date $t+s$ depends *only* on y, x and s and *not* on t.

In order to have a model that allows for present bias, we therefore need to model discounting in a different way. One simple modification, popularized by Laibson [1997], is known as *quasi-hyperbolic discounting* this replaces the exponential formulation of equation 1 with the following

$$U(c_1, c_2, c_3) = u(c_1) + \beta\delta u(c_2) + \beta\delta^2 u(c_3)$$

This model introduces the term β, which can be thought of as the *additional* discounting which occurs between period 1 and all subsequent periods: the effective discount rate between period 1 and period 2 is $\beta\delta$, while between any two subsequent periods is δ.

To see the implications of this change, we can revisit question 1 and 2. In the first case, the earlier choice is preferred if

$$\delta\beta \leq \frac{u(2)}{u(3)}$$

while in the second case the earlier choice is still preferred if

$$\delta \leq \frac{u(2)}{u(3)}$$

Thus present bias is possible if

$$\delta\beta < \frac{u(2)}{u(3)} < \delta$$

which is possible as long as $\beta < 1$.

2.3 Discounting and Time Consistency

Informally, I have argued that present bias represents a form of temptation by which people overweight current consumption. Separately, I have argued that preference for commitment is a good way to spot temptation problems. A natural question is whether there is a theoretical link between the two. The answer is yes, and is related to the concept of time consistency.

To understand this concept, let's think again about a decision maker who lives for three periods (which we will indicate by $t = 1, 2, 3$). In each period, the DM receives an amount of money y_t (for convenience, lets assume that this amount of money is the same amount y in each period). They can they either consume this income (c_t) and receive utility $u(c_t)$, or save it (to make things simple lets assume that the interest rate is equal to 0).

We can think about two different ways of solving this model. First where the decision maker at period 1 gets to fix (i.e. commit) to consumption in each of the three periods. In other words, the DM solves the problem

$$\begin{aligned} & \max_{c_1,c_2,c_3} u(c_1) + \beta\delta u(c_2) + \beta\delta^2 u(c_3) \\ \text{subject to } c_1 + c_2 + c_3 &= 3y \end{aligned}$$

We can solve this problem by setting up the legrangian

$$\begin{aligned} & u(c_1) + \beta\delta u(c_2) + \beta\delta^2 u(c_3) \\ & -\lambda(c_1 + c_2 + c_3 - 3y) \end{aligned}$$

which gives the following equations

$$\begin{aligned} u'(c_1) &= \lambda \\ \beta\delta u'(c_2) &= \lambda \\ \beta\delta^2 u'(c_3) &= \lambda \end{aligned}$$

For simplicity, we will assume Constant Relative Risk Aversion (CRRA) utility, in which $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$.. Note that this function is ill defined for $\sigma = 1$. In fact, it turns out that the limit of this utility function whethe $\sigma \to 1$ is the log utility function. Something to keep in mind. Note also that we have $u'(c) = c^{-\sigma}$, and so our Legrangian gives us

$$c_1^{-\sigma} = \beta\delta c_2^{-\sigma} = \beta\delta^2 c_3^{-\sigma}$$

and so

$$c_1 + (\beta\delta)^{\frac{1}{\sigma}} c_1 + \left(\beta\delta^2\right)^{\frac{1}{\sigma}} c_1 = 3y$$

or

$$\begin{aligned} c_1 &= \frac{3y}{1 + (\beta\delta)^{\frac{1}{\sigma}} + \left(\beta\delta^2\right)^{\frac{1}{\sigma}}} \\ c_2 &= (\beta\delta)^{\frac{1}{\sigma}} c_1 \\ c_3 &= \left(\beta\delta^2\right)^{\frac{1}{\sigma}} c_1 \end{aligned}$$

Next let's think of the decision maker as consisting of three different 'selves' that live at time 1 2 and 3. The decision maker at time 3 decides how much to consume at time 3, the DM at time two decides consumption at time 2 and so on. Anything that is not consumed is handed on to the next 'self'. Let w_2 be the amount handed on to self 2 and w_3 the amount handed on to self 3. The utility for the period three agent is given by

$$u(c_3)$$

from period two is

$$u(c_2) + \beta\delta u(c_3)$$

and from period 1 is

$$u(c_1) + \beta\delta u(c_2) + \beta\delta^2 u(c_3)$$

Note the difference between the two approaches. In the first case, the decision maker in period 1 was boss: they decided everything. In the second case, each person gets to make their own decisions.

We can then solve this problem backwards. In the last period, the consumer will obviously just eat whatever it is they have left. Thus, if they arrive in the last period with wealth level w_3, their consumption will be

$$c_3 = w_3 + y$$

and their utility

$$u(w_3 + y)$$

Now let's think about the problem at time 2, with the DM arriving with savings level w_1. The DM will choose a consumption level c_2 in order to maximize

$$u(c_2) + \beta\delta u(c_3)$$

subject to

$$\begin{aligned} c_3 &= w_3 + y = (w_2 + y - c_2) + y_3 \\ &= w_2 + 2y - c_2 \end{aligned}$$

Solving this problem gives

$$u'(c_2) = \beta\delta u'(c_3)$$

Some handy algebra gives us that

$$\begin{aligned} c_2 &= \frac{w_2 + 2y}{1 + (\beta\delta)^{\frac{1}{\sigma}}} \\ c_3 &= (\beta\delta)^{\frac{1}{\sigma}} \frac{(w_2 + 2y)}{1 + (\beta\delta)^{\frac{1}{\sigma}}} \end{aligned}$$

Whcih implies that

$$c_3 = (\beta\delta)^{\frac{1}{\sigma}} c_2$$

Let's compare this solution to the commitment case above. There, we showed that $c_3 = \delta^{\frac{1}{\sigma}} c_2$. First, notice that something extraordinary happens if $\beta = 1$: the two solutions are the same! It doesn't matter if the decision maker plans what to do at $t = 1$, or makes their mind up as they go along - their choices will be the same. This is the property of time consistency - a decision maker at time $t = 2$ will want to stick to plans made at $t = 1$

However it turns out that time consistency is only a property of exponential discounting. If $\beta \neq 1$, then the two solutions are different. The decision maker in period 1 will want consumption to be higher in period 3 relative to period 2 than does the decision maker is period 2. Put another way, the decision maker in period 2 would want to deviate from the plan devised by the decision maker in period 1. The decision maker in period 1 would like to force the period 2 self to set $c_3 = \delta^{\frac{1}{\sigma}} c_2$. This provides us with an important link between present bias and preference for commitment

$$\begin{aligned} & \text{Present bias} \\ \Longleftrightarrow \; & \text{non-exponential discount} \\ \Longleftrightarrow \; & \text{Time Inconsistency} \\ \Longleftrightarrow \; & \text{Preference for commitment} \end{aligned}$$

So how will the agent in period 1 behave. Here we face a choice about whether the 1st period agent is sophisticated, or if they are naive. If they are sophisticated, they know that they are going to be tempted in period two, and will take this into account in period 1. If they are naive, they will assume that the period 2 they will not suffer from temptation and will stick with whatever plans they have in period 1. Lets look at both case.

If the decision maker is sophisticated, then they know how they will behave in period 2, and so in period 1 will maximize

$$u(c_1) + \beta\delta u\left(\frac{3y - c_1}{1 + (\beta\delta)^{\frac{1}{\sigma}}}\right) + \delta^2 \beta u\left((\beta\delta)^{\frac{1}{\sigma}} \frac{3y - c_1}{1 + (\beta\delta)^{\frac{1}{\sigma}}}\right)$$

Taking first order conditions gives

$$u'(c_1) = \frac{\beta\delta}{1 + (\beta\delta)^{\frac{1}{\sigma}}} u'\left(\frac{3y - c_1}{1 + (\beta\delta)^{\frac{1}{\sigma}}}\right) + \delta^2 \beta . \frac{(\beta\delta)^{\frac{1}{\sigma}}}{1 + (\beta\delta)^{\frac{1}{\sigma}}} u'\left((\beta\delta)^{\frac{1}{\sigma}} \frac{3y - c_1}{1 + (\beta\delta)^{\frac{1}{\sigma}}}\right)$$

Giving

$$\begin{aligned} c_1^{-\sigma} \;\; = \;\; & \frac{\beta\delta}{1 + (\beta\delta)^{\frac{1}{\sigma}}} \left(\frac{3y - c_1}{1 + (\beta\delta)^{\frac{1}{\sigma}}}\right)^{-\sigma} + \\ & \delta^2 \beta . \frac{(\beta\delta)^{\frac{1}{\sigma}}}{1 + (\beta\delta)^{\frac{1}{\sigma}}} \left((\beta\delta)^{\frac{1}{\sigma}} \frac{3y - c_1}{1 + (\beta\delta)^{\frac{1}{\sigma}}}\right)^{-\sigma} \end{aligned}$$

Rearranging this gives

$$\begin{aligned} c_1^{-\sigma} &= \left[\frac{\beta\delta}{\left(1 + (\beta\delta)^{\frac{1}{\sigma}}\right)^{1-\sigma}} + \frac{\delta\left(\beta\delta\right)^{\frac{1}{\sigma}}}{\left(1 + (\beta\delta)^{\frac{1}{\sigma}}\right)^{1-\sigma}} \right] (3y - c_1)^{-\sigma} \\ &= \left[\frac{\beta\delta(1 + \left(\beta^{-1}\delta\right)^{\frac{1}{\delta}})}{\left(1 + (\beta\delta)^{\frac{1}{\sigma}}\right)^{1-\sigma}} \right] (3y - c_1)^{-\sigma} \end{aligned}$$

If we rearrange this, we get

$$\bar{c}_1 = \left[1 + \left(\frac{\beta\delta}{\left(1 + (\beta\delta)^{\frac{1}{\sigma}}\right)^{1-\sigma}} + \frac{\delta\left(\beta\delta\right)^{\frac{1}{\sigma}}}{\left(1 + (\beta\delta)^{\frac{1}{\sigma}}\right)^{1-\sigma}} \right)^{\frac{1}{\sigma}} \right]^{-1} 3y$$

Compare this to the commitment solutuon

$$c_1 = \left(1 + (\beta\delta)^{\frac{1}{\sigma}} + \left(\beta\delta^2\right)^{\frac{1}{\sigma}}\right)^{-1} 3y$$

Clearly, in general, the sophisticated agent will consume a different amount in the first period than they will if the could commit. It is difficult to see from these two equations whether or not subjects will consume more or less than the commitment case. In fact, it can go either way, depending on whether σ is greater or less than 1. The knife edge case in which it does not matter is equivalent to the case of log utility. Thus, for a sophisticated agent we know that

- first period consumption will generally be different to the commitment case, but can be higher or lower
- The ratio of second period to third period consumption will be higher than in the commitment case.
- There is a preference for commitment - i.e. they would pay to be able to choose period 2 and 3 consumption as well as period 1 consumption

Now let's consider the naive agent. The will assume that the agent in period 2 will stick to their plan, and so will behave as if they have commitment. Thus, in the first period they will act exactly like the commitment agent and consume

$$c_1 = \frac{3y}{1 + (\beta\delta)^{\frac{1}{\sigma}} + \left(\beta\delta^2\right)^{\frac{1}{\sigma}}}$$

However, in the second period, the agent will not stick to their plan. The naive agent assumed that the ratio of consumption in the second and third period would be $\delta^{\frac{1}{\sigma}}$, whereas in fact the second period agent will reoptimize and consume in the ratio of $(\beta\delta)^{\frac{1}{\sigma}}$ - i,e. will consume more in period 2 and less in period 3 than the naive agent expected. Thus we know that, for the naive agent

- First period consumption will be the same as the commitment case
- Second period consumption will be greater than the commitment case
- Third period consumption will be less than the commitment case.
- No preference for commitment.

It is worth noting that the quasi-hyperbolic discounting model is not the only one that can cause preference reversals. Imagine that you thought that anything that you were offered today, you were certain to get, while anything that you are offered in the future you will only get with probability $(1-\varepsilon)$. Even an exponential discounter could then exhibit preference reversals. Choosing the 1 today over the 1+R in period two implies

$$\begin{aligned}
u(2) + (1-\varepsilon)(\delta u(1) + \delta^2 u(1)) &> u(1) + (1-\varepsilon)(\delta u(2+R) + \delta^2 u(1)) \\
&\Rightarrow u(2) + (1-\varepsilon)\delta u(1) > u(1) + (1-\varepsilon)\delta u(2+R) \\
&\Rightarrow \frac{u(2)-u(1)}{u(2+R)-u(1)} > \delta(1-\varepsilon)
\end{aligned}$$

while choosing the 1+R units in three days time over 1 unit in two days time, this would imply

$$\begin{aligned}
u(1) + (1-\varepsilon)\left(\delta u(2) + \delta^2 u(1)\right) &< u(1) + (1-\varepsilon)\left(\delta u(1) + \delta^2 u(2+R)\right) \\
&\Rightarrow u(2) + \delta u(1) < u(1) + \delta u(2+R) \\
\frac{u(2)-u(1)}{u(2+R)-u(1)} &< \delta
\end{aligned}$$

Which is again possible if $\varepsilon > 0$