Rational Inattention Lecture 2

Mark Dean

Behavioral Economics G6943
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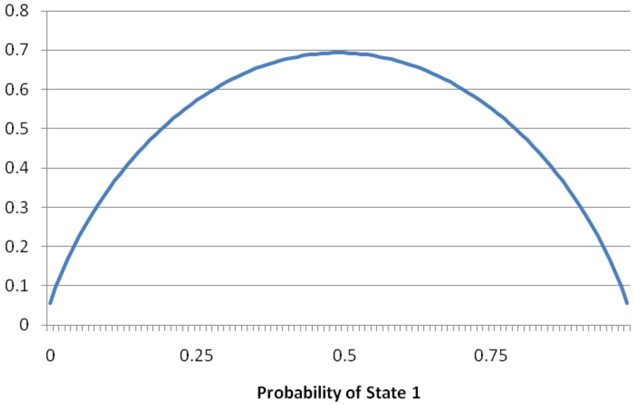
Rational Inattention and Shannon Information Costs

- We have so far considered what we can say when we are agnostic about information costs
- We now move consider behavior under a specific assumed cost for information
- Based on the concept of Shannon Entropy
  - Extremely popular in the applied literature
  - Consider this the 'Cobb Douglas' case to last week's 'revealed preference' treatment
- Long history of research in information theory
  - Quite a lot is known about how these costs behave
  - Cover and Thomas is a great resource

- Shannon Entropy is a measure of how much 'missing information' there is in a probability distribution
- In other words - how much we do not know, or how much we would learn from resolving the uncertainty
- For a random variable X that takes the value x_i with probability $p(x_i)$ for $i = 1...n$, defined as

$$\begin{aligned} H(X) &= E(-\ln(p(x_i))) \\ &= -\sum_i p(x_i) \ln(p_i) \end{aligned}$$



- Can think of it as how much we learn from result of experiment

Justification for Shannon Entropy

- Say we want our measure of entropy to have the following features
- Depends only on the probability distribution
  - $H(X) = H(p)$

Justification for Shannon Entropy

- Say we want our measure of entropy to have the following features
- Depends only on the probability distribution
- Maximized at a uniform probability distribution
    - $\max_{p \in \Delta^M} H(p) = H\left(\left\{\frac{1}{M}, \frac{1}{M}, ..., \frac{1}{M}\right\}\right)$

Justification for Shannon Entropy

- Say we want our measure of entropy to have the following features
- Depends only on the probability distribution
- Maximized at a uniform probability distribution
- Unaffected by adding zero probability state
  - $H(\{p_1....p_M\}) = H(\{p_1....p_M, 0\})$

Justification for Shannon Entropy

- Say we want our measure of entropy to have the following features
- Depends only on the probability distribution
- Maximized at a uniform probability distribution
- Unaffected by adding zero probability state
- Additive
  - $H(X, Y) = H(X) + \sum_x p(x) H(Y|x)$
  - How much you learn from observing X, plus how much you additionally learn from observing Y
  - Implies that the entropy of two independent variables is just $H(X) + H(Y)$
  - 'Constant returns to scale' assumption

Justification for Shannon Entropy

- Say we want our measure of entropy to have the following features
- Depends only on the probability distribution
- Maximized at a uniform probability distribution
- Unaffected by adding zero probability state
- Additive
- Then Entropy must be of the form (Khinchin 1957)

$$H(X) = -\sum_i p(x_i)\ln(p_i)$$

- Note, other entropies are available! e.g. Tsallis

$$\frac{k}{q-1}(1-\sum_i p(x_i)^q)$$

Entropy and Information Costs

- Related to the notion of entropy is the notion of Mutual Information

$$I(X, Y) = \sum_x \sum_y p(x, y) \log \frac{p(x, y)}{p(x)p(y)}$$

- Measure of how much information one variable tells you about another
- Note that $I(X, Y) = 0$ if X and Y are independent

- Note also that mutual information can be rewritten in the following way

$$
\begin{aligned}
I(X,Y) &= \sum_x \sum_y p(x,y) \log \frac{p(x,y)}{p(x)p(y)} \\
&= \sum_x \sum_y p(x,y) \log \frac{p(x|y)}{p(x)} \\
&= \sum_y \sum_x p(x,y) \ln P(x|y) - \sum_x \sum_y p(x,y) \ln p(x) \\
&= \sum_y p(y) \sum_x p(x|y) \ln P(x|y) - \sum_y p(x) \ln p(x) \\
&= H(X) - E(H(X|Y))
\end{aligned}
$$

- Difference between entropy of X and the expected entropy of X once Y is known

Mutual Information and Information Costs

- Mutual Information between states and signals often used to model information constraints
- Sims [2003] focused on a hard constraint on the amount of entropy a DM can use
- We will start by focussing on the case of **costs that are linear in mutual information**

$$
\begin{aligned}
K(\mu,\pi) &= \lambda(H(\mu) - E\left(H(\gamma)\right)) \\
&= \lambda\begin{pmatrix} \sum_{\gamma\in\Gamma(\pi)}\pi(\gamma)\sum_{\Omega}\gamma\left(\omega\right)\ln\gamma(\omega) \\ -\sum_{\Omega}\mu(\omega)\ln\mu\left(\omega\right) \end{pmatrix}
\end{aligned}
$$

- For convenience use γ to refer to the posterior beliefs generated by signal γ

Mutual Information and Information Costs

- Can be justified by information theory
- Say you are going to observe n repetitions of the state Ω (let ω^n be a typical element)
- You are allowed to send a message consisting of nR bits (R is the rate)
- Decoded in order to generate n repetitions of the signal space Γ (let γ^n be a typical element)
- Define $d(\omega, \gamma)$ be the loss associated with receiving signal γ in state ω, and $\hat{d}(\omega^n, \gamma^n) = \frac{1}{n} \sum d(\omega_i^n, \gamma_i^n)$

- Rate Distortion Theorem: Let $R(D)$ be the minimal rate needed to generate loss D as $n \to \infty$, then

$$R(D) = \min_{\pi \in \Pi} I(\Omega, \Gamma) \text{ s.t. } \sum_{(\gamma,\omega)} \mu(x)\pi(\gamma|x)d(\omega,\gamma) \leq D$$

- Implies (assuming strict monotonicity)

$$\min \sum_{(\gamma,\omega)} \mu(x)\pi(\gamma|x)d(\omega,\gamma) \text{ s.t. } I(\Omega,\Gamma) \leq R(D)$$

- is equivalent to

$$\min \sum_{(\gamma,\omega)} \mu(x)\pi(\gamma|x)d(\omega,\gamma) \text{ s.t. } R \leq R(D)$$

- See Cover and Thomas Chapter 10.

- Key feature: Entropy is strictly *concave*
- So negative of entropy is strictly convex
- Say we choose a signal structure with two posteriors γ and γ'
- It must be that

$$P(\gamma)\gamma + P(\gamma')\gamma' = \mu$$

- so

$$\begin{aligned} P(\gamma)H(\gamma) + P(\gamma')H(\gamma') &< H(P(\gamma)\gamma + p(\gamma')\gamma') \\ &= H(\mu) \end{aligned}$$

- So the cost of 'learning something' is always positive

Solving Rational Inattention Models

- Solving the Shannon model can be difficult analytically
  - Though easier than many other models
- General approach - ignore choice of information structure, instead focus on joint distribution of choice variable and state
  - i.e. choose state dependent stochastic choice directly
  - Can do this because optimal strategy will always be 'well behaved'
  - Each action taken in at most one state
- Example (Matejka and McKay 2015) - continuous state space, finite action space
- We will talk about analytical approaches
  - Alternative, algorithmic approaches
  - e.g. Blahut-Arimotio algorithm
  - See Cover and Thomas (page 191)

Solving Rational Inattention Models

- $\mathcal{P}$ set of all state contingent stochastic choice functions for some state space Ω and set of acts A
- Remember $P(a|\omega)$ is the probability of choosing a in state ω
- Remember that, for $P \in \mathcal{P}$, the mutual information between choices a and objective state ω is given by

$$I(A, \Omega) = H(A) - H(A|\Omega)$$

Solving Rational Inattention Models

- Decision problem of agent is to choose $P \in \mathcal{P}$ to maximize

$$\sum_{a \in A} \int_{\omega} u(a(\omega)) P(a|\omega) \mu(d\omega)$$
$$-\lambda \left[\sum_{a \in A} \int_{\omega} P(a|\omega) \ln P(a|\omega) \mu(d\omega) + \sum_{a \in A} P(a) \ln P(a) \right]$$

- Subject to

$$\sum_{a \in A} P(a|\omega) = 1 \text{ Almost surely}$$

- Where $P(a)$ is the unconditional probability of choosing a
- Note another constraint which we will ignore for now

$$P(a|\omega) \geq 0 \ \forall \ a, \omega$$

The Lagrangian Function

$$\sum_{a\in A}\int_{\omega} u(a(\omega))P(a|\omega)\mu(d\omega)$$
$$-\lambda\left[\sum_{a\in A}\int_{\omega} P(a|\omega)\ln P(a|\omega)\mu(d\omega)+\sum_{a\in A}P(a)\ln P(a)\right]$$
$$-\int_{\omega}\rho(\omega)\left[\sum_{a\in A}P(a|\omega)-1\right]\mu(d\omega)$$

- $\rho(\omega)$ Lagrangian multiplier on the condition that $\sum_{a\in A} P(a|\omega)=1$
- FOC WRT $P(a|\omega)$ (assuming $>$0)

$$u(a(\omega))-\rho(\omega)+\lambda[\ln P(a)+1-\ln P(a|\omega)-1]=0$$

- Note that this is a convex problem

Solution

- FOC WRT $P(a|\omega)$ (assuming >0)

$$u(a(\omega)) - \rho(\omega) + \lambda[\ln P(a) + 1 - \ln P(a|\omega) - 1] = 0$$

- Which gives

$$P(a|\omega) = P(a) \exp^{\frac{u(a(\omega)) - \rho(\omega)}{\lambda}}$$

- Plug this into

$$\begin{aligned}\sum_{a' \in A} P(a'|\omega) &= 1 \\ \Rightarrow \quad \exp^{\frac{\rho(\omega)}{\lambda}} &= \sum_{a' \in A} P(a') \exp^{\frac{u(a'(\omega))}{\lambda}}\end{aligned}$$

- Which in turn gives...

Comments

$$P(a|\omega) = \frac{P(a) \exp^{\frac{u(a(\omega))}{\lambda}}}{\sum_{c \in A} P(c) \exp^{\frac{u(c(\omega))}{\lambda}}}$$

- Similar in form to logistic random choice
- If alternatives are ex ante identical, this *is* logistic choice
- Otherwise choice probabilities are 'warped' by $P(a)$ - which contains information on the prior value of each option
  - Important: note that $P(a)$ is endogenous, **not** a parameter
- As costs go to zero, deterministically pick best option in that state
- As costs go to infinity, deterministically pick the best option ex ante

Comments

- The MM conditions ignore the constraint

$$P(a|\omega) \geq 0 \ \forall \ a, \omega$$

- Need to know which acts will be chosen with positive probability
- Typically there will be many acts not chosen at the optimum (Jung et al. 2015)
- There will be many solutions to the necessary conditions
- Ideally, would like necessary and sufficient conditions

- Let $z(a,\omega)$ be 'normalized utilities'

$$z(a,\omega) = \exp\left\{\frac{u(a,\omega)}{\lambda}\right\}$$

- Note that the MM conditions are

$$P(a|\omega) = \frac{P(a)z(a,\omega)}{\sum_{c\in A} P(c)z(c,\omega)}$$

Necessary and Sufficient Conditions

Theorem

P is consistent with rational inattention with mutual information costs **if and only if**

$$\begin{aligned} \sum_{\omega}\left[\frac{\mu(\omega)z(a,\omega)}{\sum_{c\in A}P(c)z(c,\omega)}\right] &\leq 1 \text{ all } a\in A \\ \sum_{\omega}\left[\frac{\mu(\omega)z(a,\omega)}{\sum_{c\in A}P(c)z(c,\omega)}\right] &= 1 \text{ all } a \text{ s.t. } P(a)>0 \end{aligned}$$

and

$$P(a|\omega)=\frac{P(a)z(a,\omega)}{\sum_{c\in A}P(c)z(c,\omega)}$$

1. Identify correct **unconditional** choice probabilities
   - Equality condition for chosen actions
   - Check inequality condition for unchosen actions
2. Read off **conditional** choice probabilities using MM conditions

Example: Finding the Good Act

- Choose from a set of goods $A = \{a_1, ..., a_N\}$
- Only one of these goods is of high quality
  - u_h utility of the high quality good
  - u_l utility of the low quality good
  - μ_i prior probability that good i is the high quality good
  - WLOG assume $\mu_1 \geq \mu_2 .... \geq \mu_N$
- Common set up in many psychology experiments

Solution

- Cutoff strategy in prior probabilities: Exists c such that
  - $\mu_i > c \Rightarrow i$ chosen with positive probability
  - $\mu_i < c \Rightarrow i$ never chosen and nothing is learned about their quality
- Endogenously form a 'consideration set'
- Let $\delta = \frac{\exp(\frac{u_h}{\lambda})}{\exp(\frac{u_l}{\lambda})} - 1$: 'additional' utility from high act
- Search the best K alternatives, where K solves

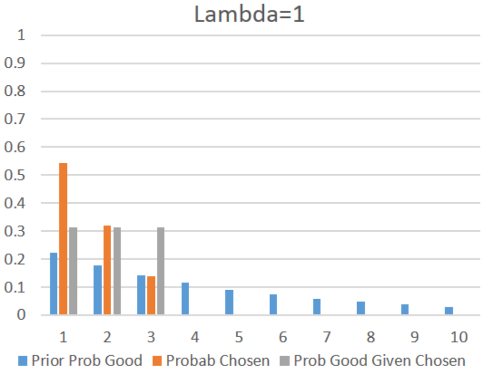
$$\mu_K > \frac{\sum_{k=1}^{K} \mu_k}{K + \delta} \geq \mu_{K+1}.$$

- Can use equality constraints to solve for unconditional choice probabilities

$$P(a_i) = \frac{\mu(\omega_i)(K+\delta) - \sum_{k=1}^{K} \mu(\omega_k)}{\delta \sum_{k=1}^{K} \mu(\omega_k)}$$

- MM conditions to solve for conditional choice probabilities

$$P(b|b = u_h) = \frac{P(b)\delta}{\sum_{c \in A} P(c)}$$

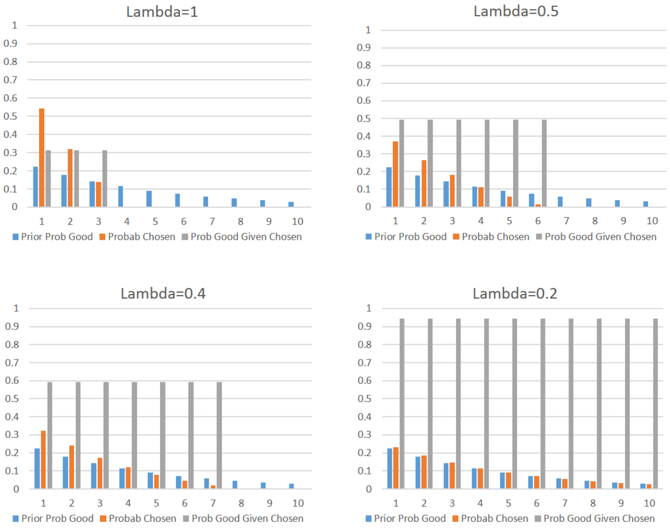


- Exponential priors
- $u_h = 1, \ u_l = 0$

Features of the Solution

- 'Consideration set' of alternatives chosen with positive probability
- Mistakes even amongst alternatives in the consideration sets
- Ex ante probability of alternative being good conditional on being chosen is same for all alternatives

Choice Probabilities - Example



Importance of Sufficient Conditions

- The MM necessary conditions could be solved for many possible 'consideration sets'
  - Choosing any option with probability 1 will solve the necessary conditions
  - For any set C with worst alternative $\mu_{\bar{C}}$ there is a solution to the necessary conditions if

$$\frac{\mu_{\bar{C}}}{\sum_{k \in C} \mu_k} > \frac{1}{|C| + \delta}.$$

- Do no reference unchosen actions
- Do not determine whether higher utility could be obtained with a different consideration sets
- This is the advantage of the sufficient conditions

The Linear Quadratic Gaussian Case

- One case in which this problem becomes more tractable is if the input and output signal are both normal
- The entropy of a normal variable $X \sim N(\mu, \sigma_x^2)$ is given by

$$H(Y) = \frac{1}{2} \ln(2\pi e \sigma_x^2)$$

- If Y and X are both normal, then

$$E(H(Y|X)) = \int_x f(x) \int_y f(y|x) \ln f(y|x) d(y) d(x)$$

- As $y|x$ is distributed normally with variance $(1-\rho^2)\sigma_y^2$, this becomes

$$\begin{aligned} E(H(Y|X)) &= \int_x f(x) \frac{1}{2} \ln(2\pi e \sigma_{y|x}^2) d(x) \\ &= \frac{1}{2} \ln(2\pi e (1-\rho^2)\sigma_y^2) \end{aligned}$$

The Linear Quadratic Gaussian Case

- As mutual information is given by

$$\begin{aligned} & H(Y) - E(H(Y|X)) \\ = \ & \frac{1}{2}\ln(2\pi e\sigma_y^2) - \frac{1}{2}\ln(2\pi e(1-\rho^2)\sigma_y^2) \end{aligned}$$

- In this case, the mutual information is given by

$$\frac{1}{2}\ln(1-\rho^2)$$

- So information costs depend only on the covariance of the two signals!
- It turns out that joint normality is optimal if the utility function is quadratic in the relationship between the objective and subjective state
    - Choice of variance on some normally distributed error term
- However, note that some papers *assume* normality (this is bad)

Set Up

- There is another way to approach this problem which possibly gives more insight
- Assume we are choosing Q, a (simple) distribution over posterior beliefs, with $Q(\gamma)$ the probability of belief γ
- We can also work with a generalized cost function

$$\sum_{\Gamma} Q(\gamma)T(\gamma) - T(\mu)$$

  where T is some strictly convex function
- For example, we could replace Shannon entropy with other types of entropy.
- Call this the class of 'posterior separable' cost functions

- One way to gain insight into what is going on is to rewrite the objective function

$$\begin{aligned} & \sum_{\Gamma} Q(\gamma)\left[\max_{a\in A}\sum_{\Omega}\gamma(\omega)u(a,\omega)\right] - \left[\sum_{\Gamma} Q(\gamma)T(\gamma) - T(\mu)\right] \\ = & \sum_{\Gamma} Q(\gamma)\left[\max_{a\in A}\sum_{\Omega}\gamma(\omega)u(a,\omega) - T(\gamma)\right] + T(\mu) \\ = & \sum_{\Gamma} Q(\gamma)\max_{a\in A} N_a(\gamma) \end{aligned}$$

- Each γ and a has a net utility associated with it

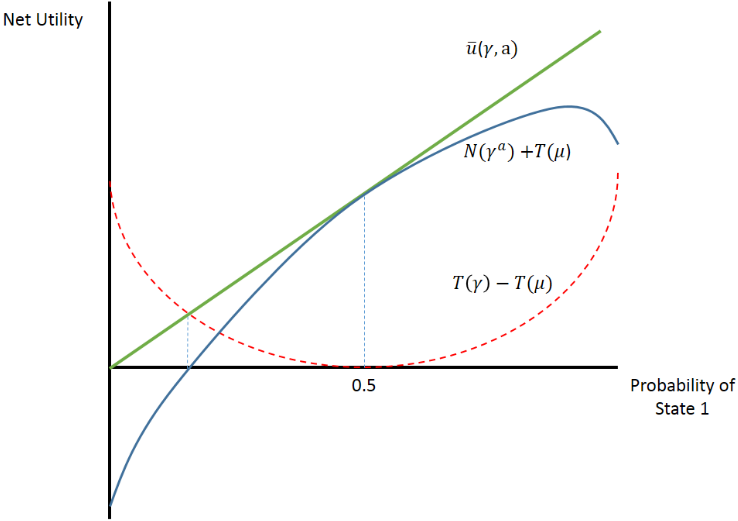
$$N_A(\gamma) = \sum_{\Omega}\gamma(\omega)u(a,\omega) - [T(\gamma) - T(\mu)]$$

- Aim is to pick distribution of posteriors which maximizes the expected value of net utilities subject to

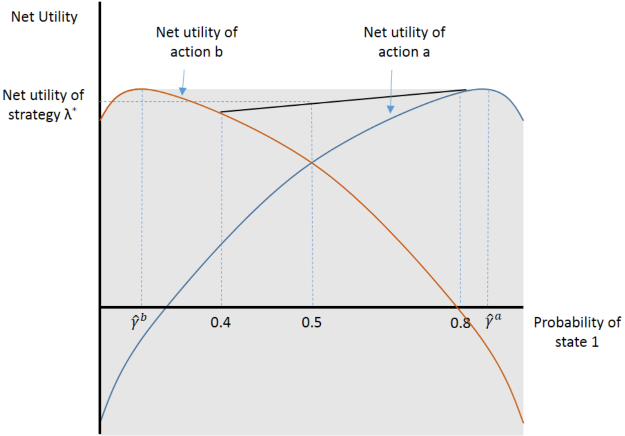
$$\sum_{\gamma\in\Gamma(\pi)} Q(\gamma)\gamma = \mu$$

- Consider a simple case with two states and two acts

Action	**Payoff in state 1**	**Payoff in state 2**
a	10	0
b	0	10

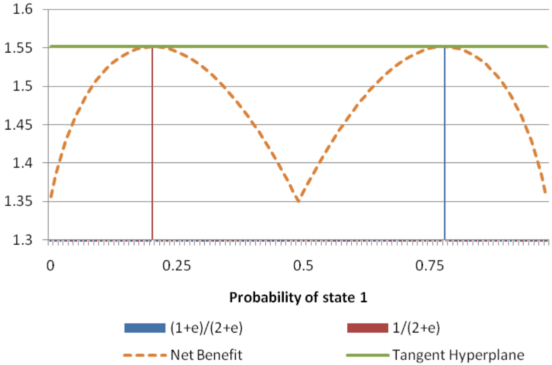

Net Utility
$\bar{u}(\gamma, \mathrm{a})$
$N(\gamma^a) + T(\mu)$
$T(\gamma) - T(\mu)$
0.5
Probability of State 1

Optimal Strategy



- What to find the posteriors which support the highest chord above the prior
- The solution for every possible prior defined by the lower epigraph of the concavified net utility function

Finding the Optimal Strategy



- Optimal posteriors identified by hyperplane that supports the set of feasible net utilities.

Theorem

Given decision problem $(\mu, A) \in \Gamma \times \mathcal{F}$ a set of posteriors are rationally inattentive if and only if:

1. **Invariant Likelihood Ratio (ILR) Equations for Chosen Acts**: *given $a, b \in B$, and $\omega \in \Omega$,*

$$\frac{\gamma^a(\omega)}{z(a(\omega))} = \frac{\gamma^b(\omega)}{z(b(\omega))}$$

2. **Likelihood Ratio Inequalities for Unchosen Acts**: *given act a chosen with positive probability and $b \in A$,*

$$\sum_{\omega \in \Omega} \left[\frac{\gamma^a(\omega)}{z(a(\omega))} \right] z(b(\omega)) \leq 1.$$

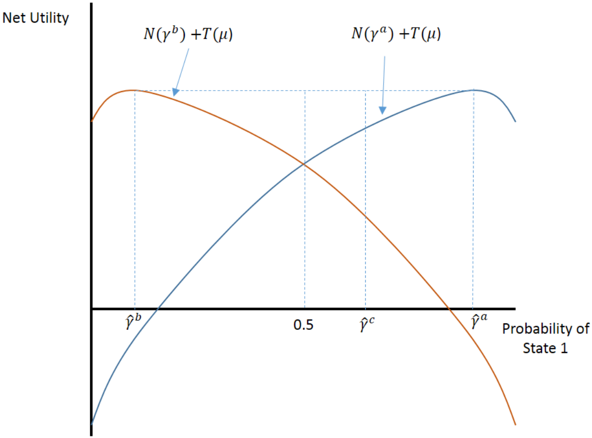
- We have necessary and sufficient conditions to characterize the Shannon model
- But these do not necessarily help us understand the behaviors that it predicts
- Might be helpful to have a more ‘behavioral’ characterization

Posterior Separability

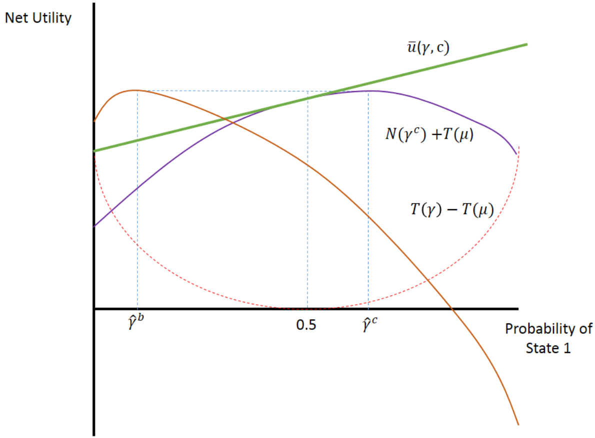
- Turns out that we can characterize using three behavioral axioms
  - Plus some technical ones that we won't bother with

1. Separability
2. Locally Invariant Posteriors
3. Invariance Under Compression

Separability



Separability



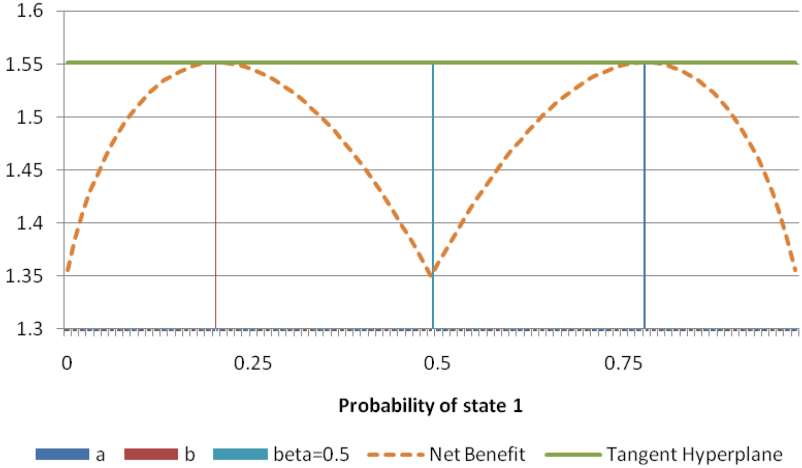
Separability

- Separability states you can always do this
  - For any set of chosen acts and associated posteriors
  - Can switch out one posterior and replace it with another posterior
  - Changing only the associated act.

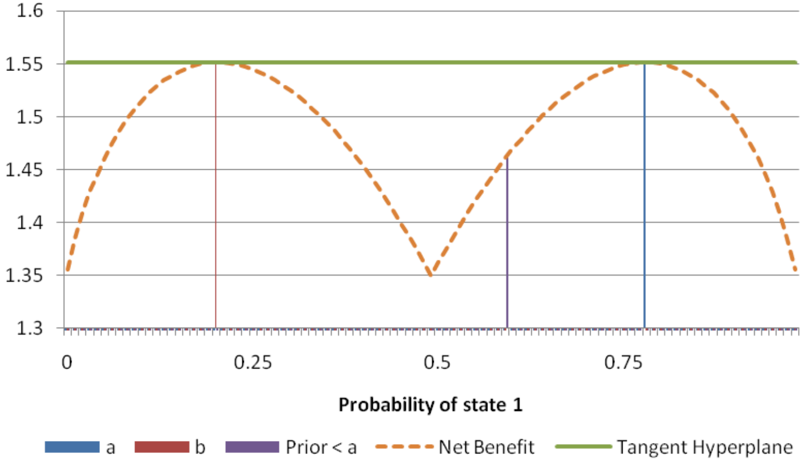
- Example: 2 states, 2 actions

Action	Payoff in state 1	Payoff in state 2
$\mathbf{f}^1$	x	0
$\mathbf{f}^2$	0	x

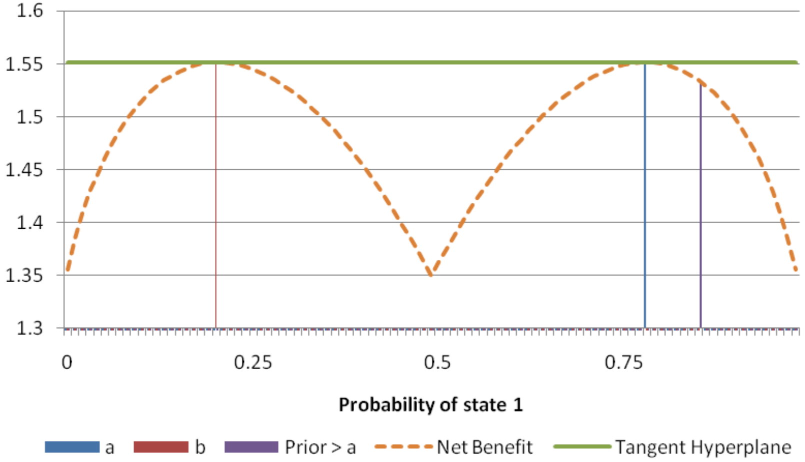
Behavior at 0.5 Prior



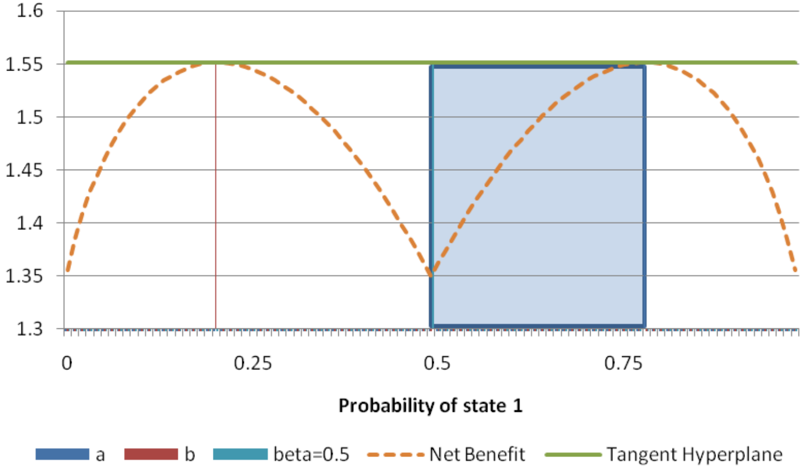
Behavior for prior<a



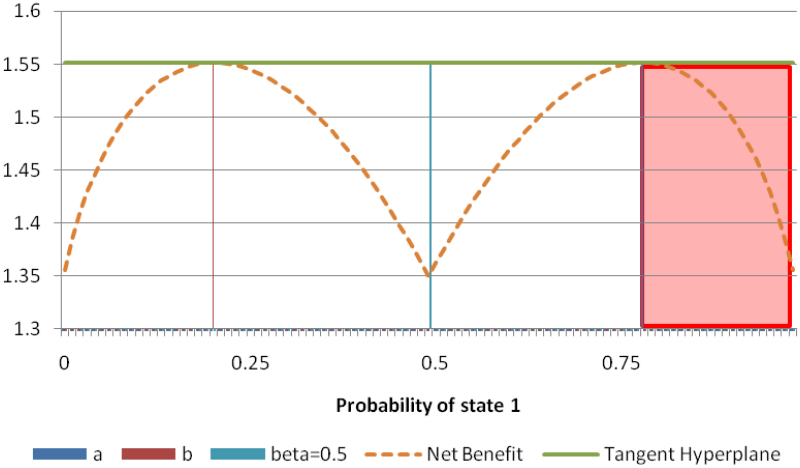
Behavior for prior>a



Same Posteriors as for 0.5 prior



No Information Gathered



Locally Invariant Posteriors

- Locally Invariant posteriors: If a set of posteriors $\{\gamma^a\}_{a\in A}$ are optimal for decision problem $\{\mu, A\}$ and are also feasible for $\{\mu', A\}$ then they are also optimal for that decision problem
- Choice probabilities move 'mechanically' with prior to maintain posteriors
- Useful in, for example, models in which consumers are rationally inattentive to quality
    - As the prior distribution of quality changes, posterior beliefs do not
    - See Martin [2014]

Invariance Under Compression

- The Shannon model is clearly 'special' in many ways in the class of UPS model
- The literature has noted many properties
  - Symmetry
  - Separability of Orthogonal Decisions
  - Lack of Complementarities
- All of these properties can be captured in a single axiom
  - Invariance Under Compression

Invariance Under Compression - An Example

- Consider decision problem (i)

State	ω_1	ω_2
Prior Prob	0.5	0.5
Payoff Action A	10	0
Payoff Action B	0	10

- And now decision problem (ii) which splits ω_2

State	ω_1	ω_2	ω_3
Prior Prob	0.5	0.2	0.3
Payoff Action A	10	0	0
Payoff Action B	0	10	10

Invariance Under Compression - An Example

- How should behavior change between the two decision problems?
- In principal, many things could happen
  - Could be harder to learn about two states that one, so less accurate in (ii) than (i)
  - Could be easier to learn about two states that one, so more accurate in (ii) than (i)
- Shannon model says that behavior should not change
  - $P_i(a|\omega_2) = P_{ii}(a|\omega_2) = P_{ii}(a|\omega_3)$

- Invariance under Compression formalizes this
- Defines the concept of a 'basic' decision problem
  - No two states have the same payoff for all acts
- Every decision problem has associated basic forms
- Choice behavior the same when moving between decision problems and their basic forms
- Corollaries
  - Behavior the same in every state which is payoff equivalent
  - Moving prior probabilities between payoff equivalent states does not change behavior

Experimental Tests

- Locally Invariant Posteriors
- Invariant Likelihood Ratio and Response to Incentives
- Invariance Under Compression

- Locally Invariant Posteriors
- Invariant Likelihood Ratio and Response to Incentives
- Invariance Under Compression

Table 1: Experiment						
Decision		Payoffs				
Problem	$\mu(1)$	$U(a(1))$	$U(a(2))$	$U(b(1))$	$U(b(2))$
1	0.50	10	0	0	10
2	0.60	10	0	0	10
3	0.75	10	0	0	10
4	0.85	10	0	0	10

- Two **unequally** likely states
- Two actions (a and b)
- 54 subjects

Prediction

- Each subject has 'threshold belief'
  - Determined by information costs
- If prior is within those beliefs
  - Both actions used
  - Learning takes place
  - Same posteriors always used
- If prior is outside these beliefs
  - No learning takes place
  - Only one action used

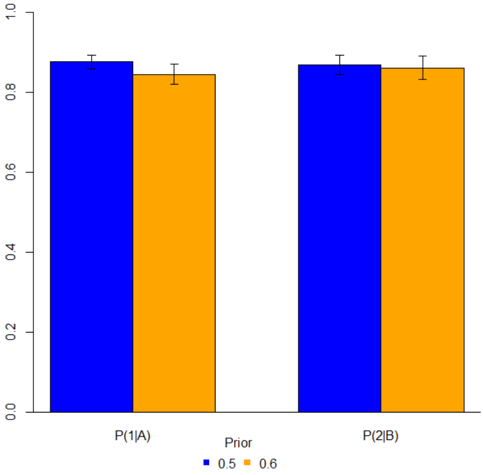
- Distribution of thresholds for 54 subjects

Posterior Range	N	%
[0.5,0.6)	14	25
[0.6,0.75)	12	22
[0.75,0.85)	12	22
[0.85,1]	16	29

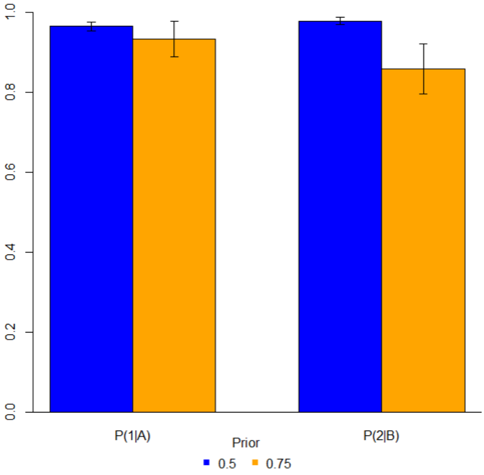
- Fraction of subjects who gather no information and always choose *a*

Table 10: Testing the 'No Learning' Prediction: Fraction of subjects who never choose b				
		$\mu(1)$		
		DP8	DP9	DP10
		0.6	0.75	0.85
Point estimates	$\gamma_7^a(1) < \mu_i(1)$	35%	27%	29%
	$\gamma_7^a(1) \geq \mu_i(1)$	0%	7%	13%
Significant differences	$\gamma_7^a(1) < \mu_i(1)$	33%	46%	41%
	$\gamma_7^a(1) \geq \mu_i(1)$	3%	10%	14%

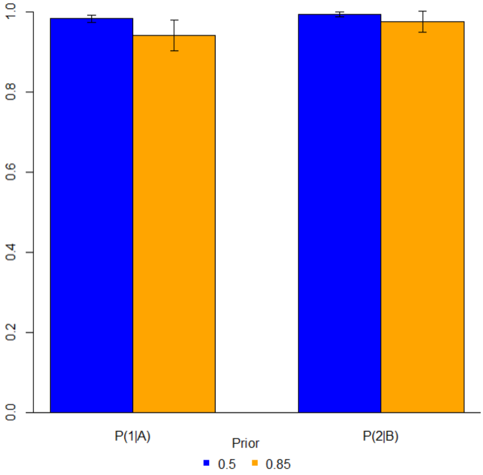
Results - Threshold Greater than 0.6



Results - Threshold Greater than 0.75



Results - Threshold Greater than 0.85



- Locally Invariant Posteriors
- Invariant Likelihood Ratio and Response to Incentives
- Invariance Under Compression

Invariant Likelihood Ratio and Responses to Incentives

- For chosen actions our condition implies

$$\frac{u(a(\omega)) - u(b(\omega))}{\ln \bar{\gamma}^a(\omega) - \ln \bar{\gamma}^b(\omega)} = \lambda$$

- Constrains how DM responds to changes in incentives

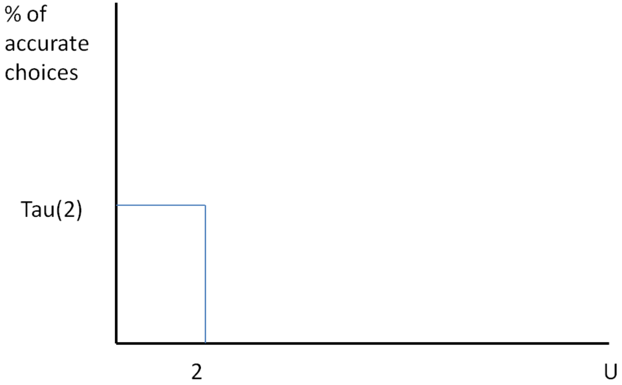
Invariant Likelihood Ratio - Example

Experiment 2				
Decision Problem	Payoffs			
	$U(a,1)$	$U(a,2)$	$U(b,1)$	$U(b,2)$
1	5	0	0	5
2	40	0	0	40
3	70	0	0	70
4	95	0	0	95

$$\frac{5}{\ln \bar{\gamma}^a(5) - \ln \bar{\gamma}^b(5)} = \frac{40}{\ln \bar{\gamma}^a(40) - \ln \bar{\gamma}^b(40)} = \ldots = \lambda$$

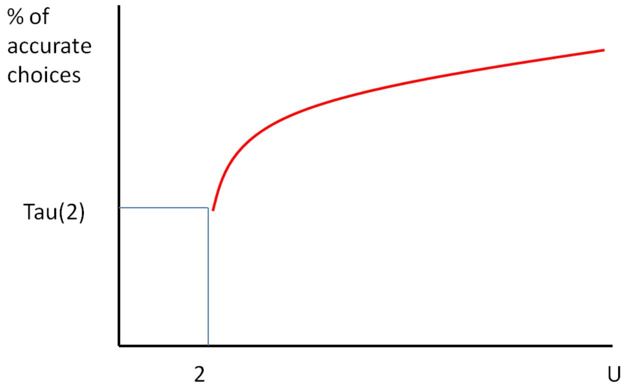
- One observation pins down λ
- Determines behavior in all other treatments

Invariant Likelihood Ratio - Example



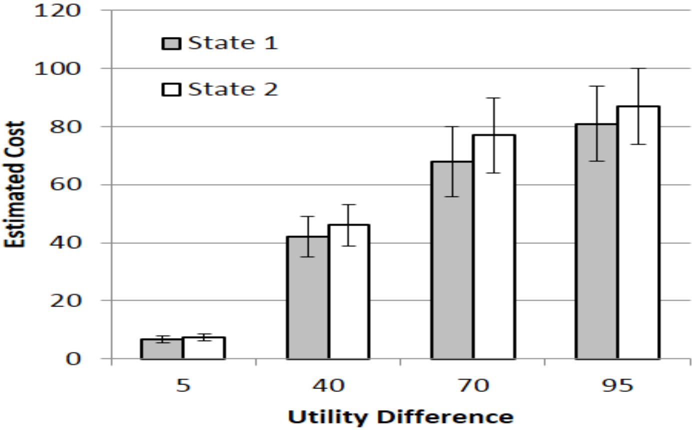
- Observation of choice accuracy for $x = 5$ pins down λ

Invariant Likelihood Ratio - Example

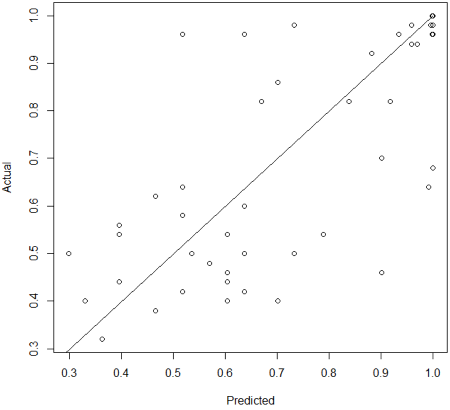


- Implies expansion path for all other values of x
- This does not hold in our experimental data

Invariant Likelihood Ratio - An Experimental Test



Individual Level Data



- Predicted vs Actual behavior in DP 4 given behavior in DP 1
- 44% of subjects adjust significantly more slowly than Shannon
- 19% significantly more quickly

- Locally Invariant Posteriors
- Invariant Likelihood Ratio and Response to Incentives
- Invariance Under Compression

Symmetry

- Compression implies the property of **symmetry**
- Behavior invariant to the labelling of states
- Optimal beliefs depend **only** on the relative value of actions in that state
- Implies that there is no concept of ‘perceptual distance’

A Simple Example

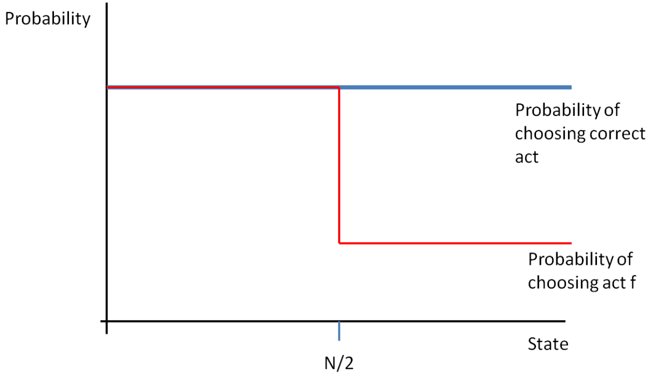
- N equally likely **states of the world** $\{1, 2....., N\}$
- Two **actions**

	Payoffs	
States	$1, \ldots \frac{N}{2}$	$\frac{N}{2} + 1, .., N$
action f	10	0
action g	0	10

- Mutual Information predicts a *quantized* information structure
  - Optimal information structure has **2 signals**
  - Probability of making correct choice is **independent of state**

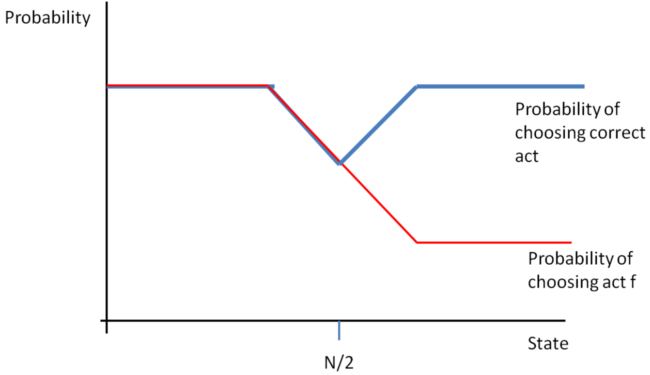
$$\frac{\exp\left(\frac{u(10)}{\lambda}\right)}{1 + \exp\left(\frac{u(10)}{\lambda}\right)}$$

Predictions for the Simple Problem - Shannon



- Probability of correct choice does not go down near threshold

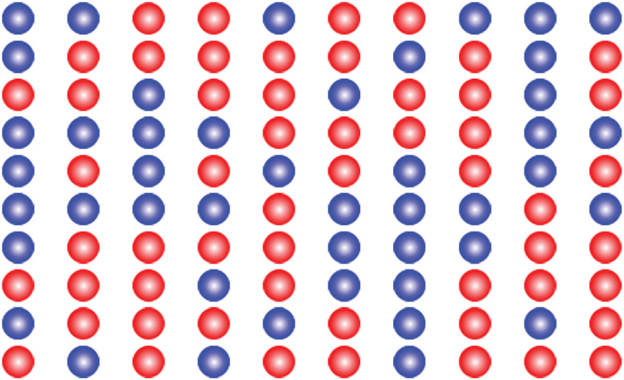
Predictions for the Simple Problem - Shannon



- Not true of other information structures (e.g. uniform signals)

Symmetry

- Shannon Model makes strong predictions for the simple problem
  - Accuracy not affected by closeness to threshold
  - In contrast to (e.g.) uniform signals
- Which model is correct?
  - It may depend on the **perceptual environment**
- Test prediction in two different environments



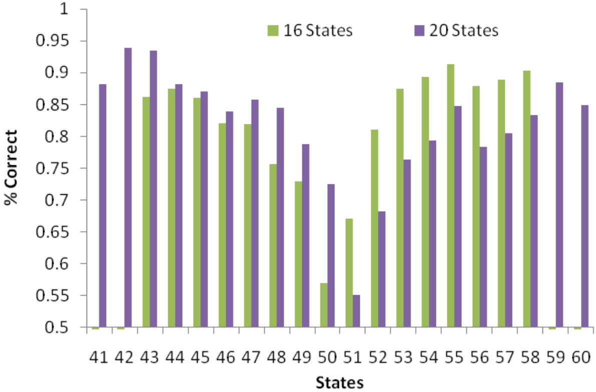
Action	**Payoff $\leq$ 50 Red**	**Payoff $>$ 50 Red**
f	10	0
g	0	10

Environment 2 (Letters)

J	P	P	J	J	L
P	N	K	N	K	M
J	Q	M	O	L	O
O	M	L	N	Q	J
Q	K	J			

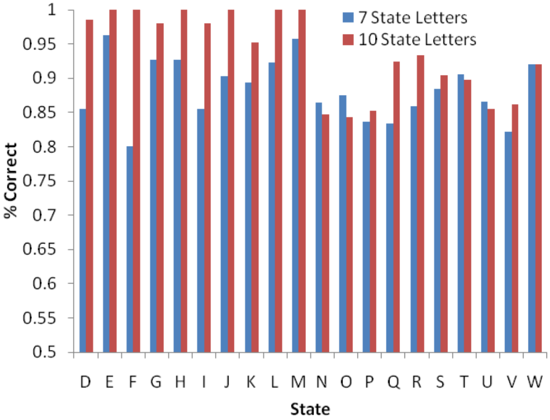
Action	**Payoff state letter $<$ N**	**Payoff state letter $\geq$ N**
f	10	0
g	0	10

- 2 treatments
- 'Balls' Experiment
  - 23 subjects
  - Vary the number of states
- 'Letters' Experiment
  - 24 subjects
  - Vary the relative frequency of the state letter
- Test whether probability of correct choice is lower nearer the threshold



- Probability of correct choice significantly correlated with distance from threshold ($p<0.001$)

Letters Experiment

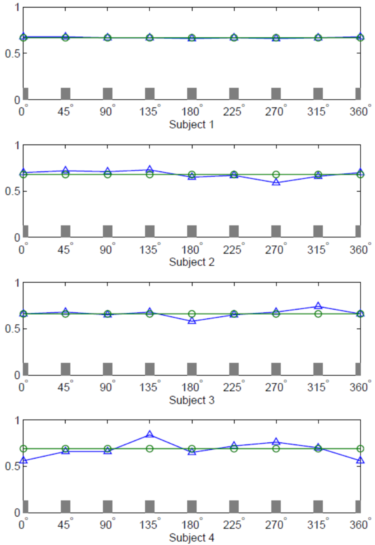


- Probability of correct choice does vary between states
- But is not correlated with distance from threshold (p=0.694)

Invariance Under Compression

- Another failure of Invariance Under Compression comes from Shaw and Shaw [1977]
- Have to recognize which of three letters has appeared
- Letter can appear at any of 8 points in a circle
- Each appearance point equally likely
- Have to say what letter appeared
- Note that the position in which the letter appears is payoff irrelevant

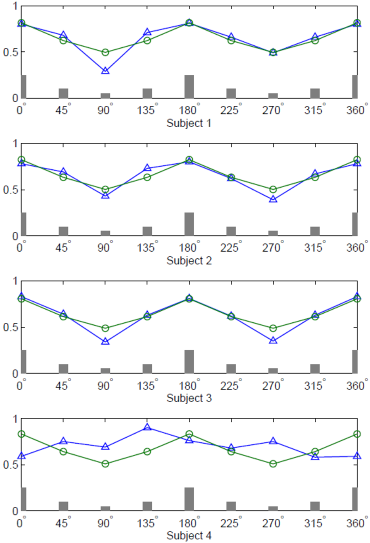
Further Prior Invariance



Further Prior Invariance

- Now make it more likely that letter appears at 'Due North' or 'Due South'
- Changes priors across payoff irrelevant states
- Should not affect behavior

Further Prior Invariance



Can we Improve on Shannon?

- These experiments tested three key properties of Shannon
  - Locally Invariant Posteriors
  - Invariant Likelihood Ratio
  - Invariance Under Compression (and in particular symmetry)
- LIP did okay(ish), the others did pretty badly
  - Expansion path problem
  - Symmetry problem
- Can we modify the Shannon model to better fit this data?

- To fix the expansion path problem there are two obvious routes

1. Posterior Separable cost functions

$$K(\mu, \pi) = \sum_{\Gamma} Q(\gamma) T(\gamma) - T(\mu)$$

   - e.g. we could use Generalized Entropy

$$T_{\rho}^{Gen}(\gamma) = \begin{cases} \left(\frac{1}{(\rho-2)(\rho-1)|\Gamma|} \sum_{\Gamma} \hat{\gamma}^{2-\rho} - 1 \right) & \text{if } \rho \neq 1 \text{ and } \rho \neq 2; \\ \frac{1}{|\Gamma|} \left(\sum_{\Gamma} \hat{\gamma} \ln \hat{\gamma} \right) & \text{if } \rho = 1; \\ -\frac{1}{|\Gamma|} \left(\sum_{\Gamma} \ln \hat{\gamma} \right) & \text{if } \rho = 2. \end{cases}$$

2. Drop the assumption that costs are linear is Shannon mutual information

$$K(\mu, \pi) = \kappa \left(\sum_{\gamma \in \Gamma(\pi)} \pi(\gamma) \left[-H(\gamma) \right] - \left[-H(\mu) \right] \right)^{\sigma}$$

Symmetry

- It is fairly obvious why symmetry fails in the 'Balls' treatment
  - Nearby states are harder to distinguish than those further away
  - Shannon cannot take this into account
- Hebert and Woodford [2017] propose a solution
  - Divide the state space into I overlapping 'neighborhoods' $X_1 ... X_I$
  - An information structure is assigned a cost for each neighborhood based on the prior and posteriors conditional on being in that neighborhood
  - Total costs is the sum across all neighborhoods

$$\sum_{i=1}^{I} \mu(X_i) \sum_{\gamma} Q(\gamma|X_i) \left[-H(\gamma|X_i)\right] - \left[-H(\mu|X_i)\right]$$

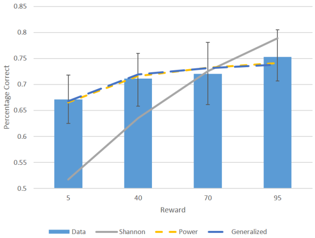
- Has a number of attractive features
  - Introduces perceptual distance to Shannon-like models
  - Qualitatively fits data from psychometric experiments
  - Can be 'microfounded' as resulting from a process of sequential information acquisition

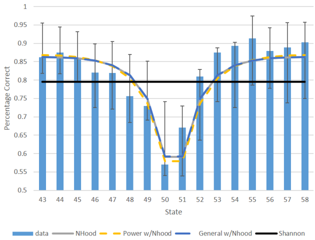
Applying Alternative Cost Functions

- We can combine these ideas to come up with a family of cost function to estimate

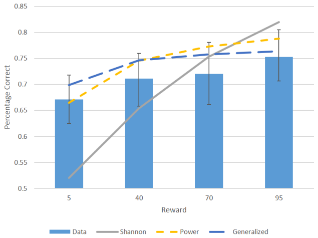
1. Linear mutual information with neighborhoods
   - Assume one global neighborhood, plus one neighborhood for each sequential pair of states
   - Cost within each neighborhood based on mutual information
   - Two parameters:
     - κ_g: marginal cost of information for the global neighborhood
     - κ_l: marginal cost of information for each of the local neighborhoods
2. Non-linear mutual information with neighborhoods
   - As (1), but costs raised to a power
   - Introduces one new parameter σ
3. General mutual information with neighborhoods
   - As (1) but mutual information replaced with expected change in generalized entropy
   - Introduces one new parameter ρ

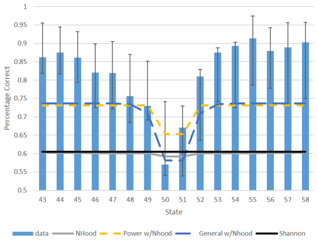
Fitted Values (Estimated Separately on Each Experiment)





Fitted Values (Estimated Jointly)





Summary

- Introduced Shannon Mutual Information as a potential cost function
  - Popular in the literature
  - 'Cobb Douglas' vs 'Revealed Preference'
- Introduced some analytical tools to help solve the Shannon model
  - MM - necessary conditions
  - Necessary + Sufficient Conditions
  - Posterior-based approach
  - Behavioral characterization
- Shown that the Shannon model can give rise to endogenous consideration set formation
- Discussed the experimental evidence for other behavioral implications
- Introduced variants of the Shannon model that better fit the data