Order Theory

Mark Dean

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1 Introduction

We are going to begin the class with some mathematical preliminaries. First, we will review some results from order theory - the branch of maths that deals with binary relations. We are interested in some of the results that order theory has to offer because many of the representation theorems that we will be looking at involve preference relations of some form or another, and preference relations are just one particular type of binary relation

2 Basics

The basic unit of analysis in order theory is the binary relation. A binary relation R on a nonempty set X is a subset of $X \times X$. It is a list of ordered pairs, which we interpret as saying that, if $\{x, y\} \in R$, then x is in relation to y, whereas if $\{x, y\}$ is not in R then x is not in relation to y. For convenience, we will write xRy if $\{x, y\} \in R$ (as you should have noticed, we have also used the symbol $\succeq$ to represent a binary relation)

Definition 1 *We can divide any binary relation R into its symmetric and asymmetric elements. We define the former as*

$$xIy \text{ iff } xRy \text{ and } yRx$$

and the latter as

$$xPy \text{ iff } xRy \text{ but not } yRx$$

Note that $R = P \cup U$. *(note that up till now we have been using* $\succ$ *and* $\sim$ *to denote the asymmetric and symmetric portions of* $\succeq$ *respectively)*

Here are some properties that binary relations can have

Definition 2 *The following are properties that a binary relation may have*

- *Reflexivity:* $xRx \; \forall \; x \in X$
- *Symmetry:* xRy *implies* yRx
- *Antisymmetric:* $xRyRx$ *implies that* $x = y$
- *Asymmetric: if* xRy *then not* yRx
- *Transitive:* $xRyRz$ *implies* xRy
- *Complete:* xRy *or* yRx
- *Acyclic:* $x_1Rx_2R...Rx_n$ *implies that* $x_1 \neq x_n$

We can use these definitions to define classes of preference relations

Definition 3 *The following are classes of preference relations*

- *If* R *is reflexive, symmetric and transitive then it is an* ***equivalence relation***
- *If* R *is reflexive and transitive then* R *is a* ***preorder*** *(what we have also been calling a preference relation)*
- *If* R *is reflexive, transitive and antisymmetric then* R *is a* ***partial order***
- *A complete partial order is a* ***linear order***

Note the difference between a preorder and a partial order. The former allows for indifferences, while the latter does not.

We call a set and a companion binary relation (X, R) a **poset** if R is a partial order, and a **loset** if R is a linear order.

An equivalence relation allows us to define the concept of equivalence classes. The equivalence class of an element x is all the elements that are in relation to x. Thus, if $\sim$ is an equivalence relation on a set X, we define the equivalence class of $x \in X$ as

$$[x]_\sim = \{y \in X | x \sim y\}$$

We can also define the quotient set set of X relative to $\sim$. This is just the set of set of equivalence classes

Definition 4 *An equivalence relation $\sim$ on a set X generates a quotient set $X/_\sim$, defined as*

$$X/_\sim = \{[x]_\sim x \in X\}$$

For any binary relation, we can define the concept of a maximal element and an upper bound:

Definition 5 *Let $\succeq$ be a binary relation on a set X: the maximal elements of set X according to $\succeq$ are defined as*

$$Max(X, \succeq) = \{x \in X | y \succ x \text{ for no } y \in X\}$$

while an upper bound is defined as

$$M(X, \succeq) = \{x \in X | x \succeq y \ \forall \ y \in X\}$$

You may be used to intuitively thinking of an upper bound and a maximal element as the same thing but this is generally not the case for arbitrary binary relations

3 Tranistive Closures and Extensions.

One issue that we are going to be interested in is whether or not a binary relation that we observe can be thought of as an incomplete glimpse of some complete preorder (why might we be interested in this?). In order to answer this question we need to introduce the concept of a transitive closure and an extension.

Definition 6 *A transitive closure of a binary relation R is a binary relation $T(R)$ that is the smallest transitive binary relation that contains R. (i.e. $T(R)$ is transitive, it contains R in the* *sense that xRy implies $xT(R)y$, and any binary relation that is smaller (in the subset sense) is either intransitive or does not contain R.)*

Transitive closures are handy things for us to work with, so it is worth describing some of their properties.

Remark 1 *Every binary relation R on any set X has a transitive closure*

Proof. *To see this, note that there is always a transitive binary relation that contains R: the trivial relation xTy for all $x, y \in X$. Furthermore, the intersection of any set of transitive relations is in itself transitive (yes?). Thus the intersection of all transitive closures that contain R always exists and is the smallest transitive relation that contains R.* ■

Remark 2 *We can alternatively define the transitive closure of a binary relation R on x as the following:*

1. *Define $R_0 = R$*
2. *Define R_m as xR_my if there exists $z_1, ..., z_m \in X$ such that $xRz_1R...Rz_mRy$*
3. *$T = R \cup_{i \in \mathbb{N}} R_m$*

Proof. *We need to show three things - that R is contained in T, that T is transitive, and that any transitive set that contains R must also contain T. The first is trivial. The second comes from the fact that, if $xTyTz$, then xR_myR_nz for some $m, n \in \mathbb{N}$. Thus, xR_pz for some $z \leq m + n$, and so xTz. Finally let G be a transitive set that contains R. We need to show that this set also contains T. Equivalently, we need to show that it contains R_i for every i. We can prove this inductively. Clearly G contains R_0, which is just R. Now assume that it contains R_n and consider x and y such that, for some $z_1, ..z_{n+1}$*

$$xRz_1R...Rz_nRz_{n+1}Ry$$

By the inductive assumption, xGz_{n+1} and also $z_{n+1}Gy$. Thus, by the transitivity of G xGy and so G contains T and we are done. ■

The next concept that we want to introduce is that of an extension

Definition 7 *Let $\succeq$ be a preorder on X. An **extension** of $\succeq$ is a preorder $\trianglerighteq$ such that*

$$\succeq \ \subset \trianglerighteq$$
$$\succ \ \subset \triangleright$$

In other words, an extension of a preorder $\succeq$ is another preorder that 'agrees' with $\succeq$ in the sense that $x \succeq y$ implies that $x \trianglerighteq y$, and $x \succ y$ implies that it is not the case that $y \trianglerighteq x$.

Of particular interest to us is going to be *complete* extensions of a preorder $\succeq$ - i.e. preorders that extend $\succeq$ and are complete. A fundamental result in order theory is that every partial order can be extended to a linear order. The is true for a preorder on any set X regardless of its cardinality.

Theorem 1 (Sziplrajn) *For any nonempty set X and partial order $\succ$ on X there exists a linear order that is an extension of $\succ$*

For cases in which X is finite, Sziplrajn's theorem can be proved pretty easily. However, to show this to be true for arbitrary sets X requires some more mathematical heavy lifting in the form of the axiom of choice. We'll discuss this proof in a minute, but before doing so, it is worth noting a couple of implications of the theorem itself First, it should be clear that the equivalant result is true for preorders

Corollary 1 *For any nonempty set X and preorder $\succeq^*$ on X there exists a complete preorder that is an extension of $\succeq^*$*

Proof. *Let $\sim$ be the symmetric part of $\succeq^*$. $\sim$ is an equivalence relation (you should check that you agree with this statement). We can use $\succeq^*$ to generate a preference relation $\trianglerighteq^*$ on the quotient set $X/_\sim$.in the following way:*

$$[x]_\sim \trianglerighteq {}^*[y]_\sim$$
$$\textit{if and only if}$$
$$x \succeq y$$

It should be obvious, (but you should check) that $\trianglerighteq^*$ is a partial order on $X/_\sim$. Thus, by Sziplrajn's theorem, there exists a linear order $\trianglerighteq$ on $X/_\sim$ that extends $\trianglerighteq^*$. Define the preference *relation* $\succeq$ *on* X *as*

$$\begin{aligned} x &\succeq y \\ &\text{if and only if} \\ [x]_\sim &\trianglerighteq [y]_\sim \end{aligned}$$

Again, you should check, but it should be obvoius that $\succeq$ *is a complete preorder that extends* $\succeq^*$

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Another handy generalization of this result is that it is not just preorders that can be extended to complete preorders. In fact, a sufficient (and necessary) condition is a variant of acyclicality I call only weak cycles

Definition 8 *Let* $\succeq$ *be a binary relation on a non-empty set* X*,* $\succ$ *be the asymmetric part of* $\succeq$ *and* $T(\succeq)$*. We say that* $\succeq$ *satisfies only weak cycles (OWC) if,*

$$xT(\succeq)y \text{ implies not } y \succ x$$

As its name suggests, the OWC condition is a weaker condition than acyclicality - it allows for cycles, but only in the symmetric part of $\succeq$. Note also that this condition is closely related to the Generalized Axiom of Revealed Preference.

Proposition 1 *Let* $\succeq$ *be a binary relation on a non-empty set* X*. Then* $\succeq$ *can be extended to a complete preorder if and only if it satisfies OWC*

Proof (OWC implies existence of complete preorder extension). *Let* $T(\succeq)$ *be the transitive closure of* $\succeq$*, with the reflexive relation on* X *added. If* $\succeq$ *satisfies OWC, then it must be the case that* $T(\succeq)$ *is an extension of* $\succeq$*. This follows directly from the OWC condition, as if* $y \succ x$ *it cannot be the case that* $xT(\succeq)y$*. Moreover* $T(\succeq)$ *is a preorder as it is transitive and reflexive. Thus by corrolary 1 there exists a complete preorder that extends* $T(\succeq)$*, and so* $\succeq$*, completing the proof.*

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Proof (Existence of complete preorder extension implies OWC). *Assume* $\succeq$ *violates OWC, so that these exists an* x *and* y *such that*

$$xT(\succeq)y \text{ and } y \succ x$$

In any extension R of $\succeq$, it cannot be the case that xRy. But therefore there is no transitive extension of $\succeq$ $\blacksquare$

4 The Axiom of Choice

In order to prove Sziplrajn's theorem, we are going to invoke the axiom of choice. This is a glimpse into some advanced set theory which we do not have the time (and I don't have the credentials) to do this subject justice, but as some of these results come up from time to time in decision theory it is worth quickly summarizing them here.

The axiom of choice is confusing, but the idea is as follows: Let $\mathcal{A}$ be an arbitrary collection of non-empty sets. What we want to know is: under what circumstances can we construct a function that selects, or 'chooses' one element from each set. In other words, under what circumstances can we construct a function

$$\begin{aligned} f &: \mathcal{A} \rightarrow \cup\mathcal{A} \\ \text{such that } f(A) &\in A\ \forall\ A \in \mathcal{A} \end{aligned}$$

In other words, if we were faced with a collection of bins, under what circumstances can we be sure that we can find a way of selecting one item from each bin.

On the face of it, it seems that the answer is that we can always do this. And in many cases, it is easy to show that this is the case. For example:

- If $\mathcal{A}$ consists of sets of natural numbers, then we could construct such a function: for example we could define a function that just takes the lowest number in each set
- If $\mathcal{A}$ consists of intervals on the real line, then we could also construct such a function, by taking the midpoint of that line
- If$\mathcal{A}$ is finite, then we may not be able to construct such a function (if we don't know anything about the things in the set), but we know that one exists. Let's ennumerate the sets as $A_1, ...A_n$. The fact that $A_1 \in \mathcal{A}$ is non-empty means that there is some element $a_1 \in A_1$. Thus we can let $f(A_1) = a_1$. By induction we can extend this principle to any arbitrary collection of n sets

The problem comes if our collection $\mathcal{A}$ both contains an infinite number of sets AND we don't know anything about the nature of each set. So, for example, if$\mathcal{A}$ is all subsets of the real line. Here, we cannot construct an explicit rule (you should try, but you won't be able to come up with a rule that you can apply to any arbitrary subset of the real line), and we can't use induction to prove the existence of such a rule[1], so how do we know that such rules exist?

The answer is that we don't: it turns out that the standard axioms of mathematics (the Zermelo-Fraenkel-Skolem axioms) do not guarantee the existence of such a rule for any arbitrary collection of non empty sets. However, it is also the case that they the universal existence of such a rule is consistent with these other axioms. Thus, the existence of such a rule is *independent* of the rest of mathematics: we can have a consistent mathematical system in which we can assume that such a rule always exists, or where we do not. In other words, we can chose to take or leave the **axiom of choice.**[2]

Definition 9 (The axiom of choice) *Let $\mathcal{A}$ be an arbitrary collection of non-empty sets. Then there exists a function*

$$\begin{aligned} f &: \mathcal{A} \to \cup\mathcal{A} \\ \textit{such that } f(A) &\in A \;\forall\; \forall \in \mathcal{A} \end{aligned}$$

Most mathematicians these days choose to take the axiom of choice, in part because it seems so intuitive. Though you do need to be careful - the axiom of choice can have some unexpected consequences. One of which (the Banach-Tarski paradox) is that, if we accept the axiom of choice then it is also the case that a solid ball in 3-dimensional space can be split into a finite number of non-overlapping pieces, which can then be put back together in a different way to yield two identical copies of the original ball. The reassembly process involves only moving the pieces around and rotating them, without changing their shape.

Ignoring this slightly worrying result, we will make use of the axiom of choice in this couse, because it will be very useful. For example the following are true if (and only if) the axiom of choice holds:

[1] We also can't just list an infinite number of statements of the form A_x is non-empty so there exist an $a_x \in A_x$ for an uncountable x, as this proof requires an infinite number of steps - which is not allowed

[2] These two results were proved by Kurt Gödel in 1939, and the second one by Paul Cohen in 1963.

1. The Cartesian product of an arbitrary number of non-empty sets is well defined

2. (Zorn's Lemma) If every loset in a given poset has an upper bound, then the poset most have a maximal element[3]

3. (The Hausdorff Maximum principle) There exists an $\supseteq$-maximal loset in every poset (i.e. there exists a loset $\{X, \succeq\}$ such that there is no other loset $\{Y, \succeq\}$ such that $Y \supset X$

We can use the last of these to prove Sziplrajn's theorem (though in fact, it should be noted, that the theorem is not equivalent to the axiom of choice)

(Sziplrajn's Theorem). Let $\succcurlyeq$ be a partial order on a non-empty set. Let T_X be the set of all partial orders that extend $\succcurlyeq$ Clearly $\{T_X, \supseteq\}$ is a poset,[4] so by the Hausforff maximum principle, it must have a maximal loset $\{A, \supseteq\}$. Define $\succcurlyeq^* = \cup A$ (i.e. the union of all the binary relations in A). Because A is a loset (and so all the sets in A can be ranked using the subset relation,) then $\succcurlyeq^*$ is a partial order that extends $\succcurlyeq$. Moreover, $\succcurlyeq^*$ is complete.To see this, assume not, then take some x, y such that neither $x \succcurlyeq^* y$ nor $y \succcurlyeq^* x$. Then, if we take the transitive closure of $\succcurlyeq^* \cup \{x, y\}$ we get an element of T_X that contains $\succcurlyeq^*$ [5] as a proper subset. But this contradicts the fact that $\{A, \supseteq\}$ is a maximal loset in $\{T_X, \supseteq\}$, a contradiction. ■

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[3] Remember that a loset is a linearly ordered set and a poset is a partially ordered set. An upper bound of a set X according to a binary relation $\succeq$ is an element

$$x \in X | x \succeq y \; \forall \; y \in X$$

A maximal element is defined as

$$x \in X | y \succ x \text{ for no } y \in X$$

[4] i.e. the relation 'is a weak subset of' is a partial order on T_X. You should check that you are happy with this statement

[5] The only tricky part here is the idea that $T(\succcurlyeq^* \cup \{x, y\})$ is an extension of $\succcurlyeq$. If not, then there must be some $w, z \in X$ such that $w \succ z$., but there is some chain

$$z \succcurlyeq^* w_1 \succcurlyeq^* w_n \succcurlyeq^* x, y \succcurlyeq^* w_m \succcurlyeq^* .. \succcurlyeq^* w_k \succcurlyeq^* w..$$

But if this was the case, then rearranging this chain gives us

$$y \succcurlyeq^* w_m \succcurlyeq^* .. \succcurlyeq^* w_k \succcurlyeq^* w. \succ z \succcurlyeq^* w_1 \succcurlyeq^* w_n \succcurlyeq^* x$$

which, as $\succcurlyeq^*$ is a transitive extension of $\succ$, would imply that $y \succcurlyeq^* x$, a contradiction.