2 Lecture 2

2.1 From Preferences to Utility Maximization?

Okay, so we have made some progress - we now know that a DM satisfies α and β if and only if we can think of them as maximizing some complete preference relation. However, our original goal was to understand how we could test whether someone looked like a utility maximizer. For this, we still have some work to do. In fact, we want to ask can we find a 'utility representation' for our binary relation, where we define a utility representation in the following way:

Definition 4 *A binary relation $\succeq$ on a set X has a utility representation if there exists a utility function $u{:}X \to \mathbb{R}$ such that*

$$\begin{aligned} u(x) &\geq u(y) \\ \text{if and only if } x &\succeq y \end{aligned}$$

for all x, $y \in X$

What we want to show is that a binary relation has a utility representation if and only if it is a complete preference relation.

Theorem 3 *Let X be a finite set. A binary relation $\succeq$ on X has a utility representation if and only if is a complete preference relation.*

Proof. *Again, we have two things to prove here, as this is an if and only if statement. Again, we will begin by showing that the axioms imply the representation, which is the more difficult direction.* ■

Proof (axioms imply representation). *We will proceed using induction on the size of the set X. That is, we will show that (i) it is true for $|X| = 1$ and (ii) if it is true for $|X| = n - 1$ then it is true for $|X| = n$. The case of $|X| = 1$ is trivial (though note that it uses reflexivity), so we will move onto the second part of the proof. Let X be a set of size n, and let $\succeq$ be a complete preference relation on X. Remove object from the set X, which we will denote x^*. Now note that X/x^* is a set of size $n-1$ and $\succeq$ induces a complete preference relation on X/x^* . Thus, there is a* *function $v : X/x^* \to \mathbb{R}$ such that $v(x) \geq v(y)$ if and only if $x \succeq y$. We will use this to construct a utility function u on X. We will set $u(x) = v(x)$ for all $x \in X/x^*$. This utility function will clearly represent $\succeq$ on X/x^* in the sense that $u(x) \geq u(y)$ if and only if $x \succeq y$ for all $x, y \in X/x^*$. Thus, all that remains to do is to is to set $u(x^*)$ and show that the utility function works here to. There are 4 cases.*

1. *$x^* \succeq \bar{x}$ and $\bar{x} \succeq x^*$ for some $\bar{x} \in X/x^*$. In this case, we set $u(x^*) = u(\bar{x})$. Now, note that, for any $y \neq x^*$*

$$\begin{aligned} u(x^*) &\geq u(y) \\ \textit{if and only if } u(\bar{x}) &\geq u(y) \\ \textit{if and only if } \bar{x} &\succeq y \\ \textit{if and only if } x^* &\succeq y \end{aligned}$$

   The third line follows from the fact that $\bar{x}$ and $y, \bar{x} \in X/x^*$, and so by the inductive hypothesis u represents the relationship between these two. The last line follows from transitivity. Using the same technique it is possible to show that $u(y) \geq u(x^*)$ if and only if $y \succeq x^*$

2. *$x^* \succeq y$ for all $y \in X$. (for the next three cases we will assume that there is no $\bar{x} \in X/x^*$ such that $x^* \succeq \bar{x}$ and $\bar{x} \succeq x^*$). In this case we set*

$$u(x^*) = \max_{y \in X/x^*} v(y) + 1$$

   Now, for any $y \neq x$, $u(x^*) > u(y)$ and $x^* \succeq y$. By assumption $y \succeq x^*$ for no $y \neq x^*$

3. *$y \succeq x^*$ for all $y \in X$. In this case, we set*

$$u(x^*) = \min_{y \in X/x^*} v(y) - 1$$

   Now, for any $y \neq x$, $u(y) > u(x^*)$ and $y \succeq x$. By assumption $x^* \succeq y$ for no $y \neq x^*$

4. *There exists at least one $y \in X/x^*$ such that $y \succeq x$ and $z \in X/x^*$ such that $x \succeq z$. In this case, define two sets: $X^* = \{x \in X/x^* | x \succeq x^*\}$ and $X_* = \{x \in X/x^* | x^* \succeq x\}$. Note that these two sets are disjoint (as we have ruled out the possibility that $x \succeq x^*$ and $x^* \succeq x$ for any $x \neq x^*$), and that, for any $x \in X^*$ and $y \in X_*$, $x \succeq y$ but not $y \succeq x$ ($x \succeq y$ follows directly*

from transitivity. If $y \succeq x$, then $x^* \succeq y \succeq x^*$, which we have ruled out by assumption). This in turn implies that

$$\min_{x\in X^*} v(x) > \max_{y\in X_*} v(y)$$

We will therefore set

$$u(x^*) = \frac{1}{2}\min_{x\in X^*} v(x) + \frac{1}{2}\max_{y\in X_*} v(y)$$

Thus, for any $x \neq x^*$

$$\begin{aligned} u(x^*) &\geq u(x) \\ \text{if and only if } x &\in X_* \\ \text{if and only if } x^* &\succeq x \end{aligned}$$

Similarly

$$\begin{aligned} u(x) &\geq u(x^*) \\ \text{if and only if } x &\in X^* \\ \text{if and only if } x &\succeq x^* \end{aligned}$$

This completes the first part of the proof. ■

Proof (Representation Implies Axioms). *This direction is relatively simple. Say that $\succeq$ is a binary relation on X and that $u : X \to \mathbb{R}$ is a utility representation of that function. Then $u(x) \geq u(x)$ implies $x \succeq x$ (reflexivity), for any x, y either $u(x) \geq u(y)$ or $u(y) \geq u(x)$ implying either $x \succeq y$ or $y \succeq x$ (completeness), and that $x \succeq y \succeq z$ implies $u(x) \geq u(y) \geq u(z)$, and so $x \succeq z$ (transitivity).* ■

Finally, note that the utility function that can represent a complete preference relation is *not* unique. It is unique only up to strictly increasing transformation. This means that, if the function u represents a set of preferences, then the function v will represent the same preferences if and only if v is a strictly increasing transform of u.

Theorem 4 *Let $u : X \to \mathbb{R}$ be a utility representation for a complete preference relation $\succeq$. Then $v : X \to \mathbb{R}$ will also represent $\succeq$ if and only if there is a strictly increasing function T such that*

$$v(x) = T(u(x)) \ \forall\ x \in X$$

Proof. *To show the if part, note that, if v is a strictly increasing transform of u then*

$$\begin{aligned} v(x) &\geq v(y) \\ \text{if and only if } u(x) &\geq u(y) \\ \text{if and only if } x &\succeq y \end{aligned}$$

To show the only if part, note that if v is not a strictly increasing transform of u, then there exists an x and y such that $u(x) > u(y)$ but $v(x) \leq v(y)$. $u(x) > u(y)$ implies that it is not the case that $y \succeq x$. Thus, v does not represent $\succeq$. ■

This uniqueness result is important, as it tells us how much information is in the utility function. In this case, it is telling us that it is only the ordinal (ordering) information that is important - that the utility number is bigger than another. The magnitude of those differences are meaningless.

Before we move on to expanding these results a little bit, it is worth thinking about what we have just done, and the benefits of this approach, specifically

1. We have started with a model that we did not know how to test because it had latent variables, and come out with two easily testable conditions for this model. These conditions are exactly equivalent to the model we started with, as they are both necessary and sufficient for utility maximization. This means that, if these conditions are satisfied, then a subject looks like a utility maximizer, while if they aren't, then the subject is not a utility maximizer. We have also managed to do this without making any assumptions about how the subject forms their utility function.

2. The representation theorem has also told us how seriously to take the concept of 'utility'. In particular, we know that utility is only defined up to strictly positive transformations. It is therefore meaningless to say things like 'the utility of x is twice that of y', because we could just as well use another utility function in which the utility of x is a million times that of y, or one where the utility of x is 1% higher than that of y. Any utility function that preserves the same ordering properties will do the job.

It will be nice for us to come up with equivalent results to this for the other behavioral models that we will come across during the course.