Consumer Choice 1

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GR6211 - Microeconomic Analysis 1

- We are now going to think a lot more about a particular type of choice we introduced two lectures ago
- **Choice from Budget Sets**
  - Objects of choice are **commodity bundles**

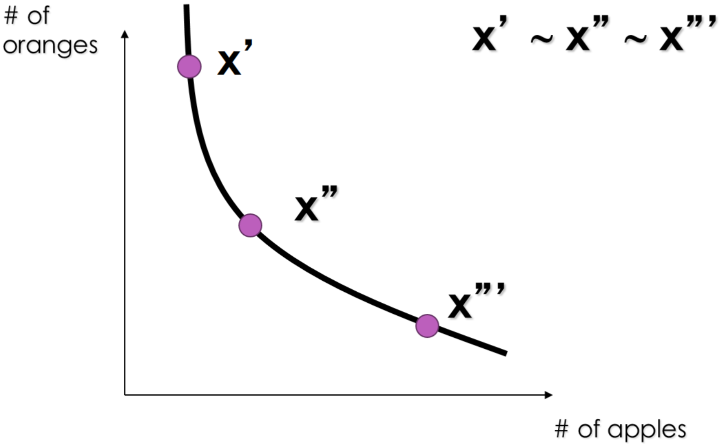
$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

  - Consumers are **price takers**
    - Treat prices and incomes as fixed
  - They can choose any bundle which satisfies their budget constraint

$$\left\{ x \in \mathbb{R}^n_+ \,|\, \sum_{i=1}^{n} p_i x_i \leq w \right\}$$

- Why are such choices so interesting?
  - Many economic interactions can be characterized this way
  - Will form the basis of the study of equilibrium in the second half of the class

- When dealing with preferences over commodity bundles it will be useful to think about **Indifference Curves**
- These are curves that link bundles that are considered indifferent by the consumer
  - i.e. the set of points $\{x | x \sim y \text{ for some } y\}$
- Useful for presenting 3 dimensional information on a two dimensional graph

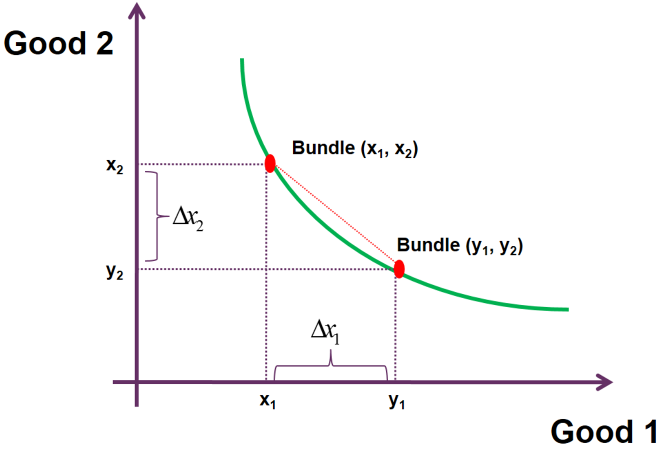


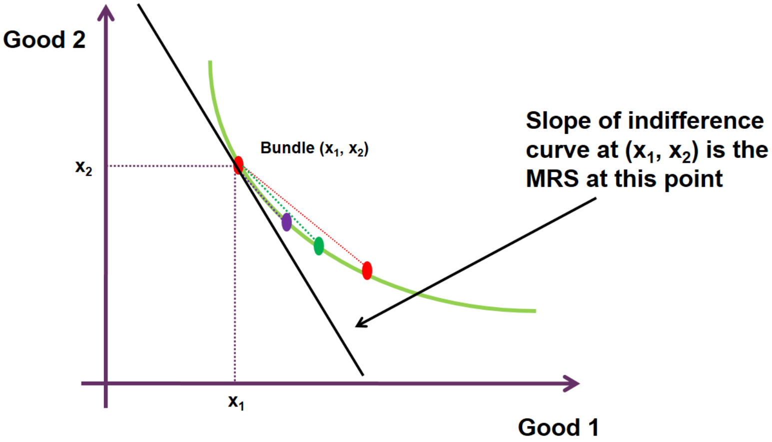
of
oranges
x'
x''
x'''
# of apples
x' ~ x'' ~ x'''

- A couple of properties of indifference curves

1. Two different indifference curves cannot cross (why?)
2. The 'slope' of the indifference curve represents the (negative of the) **marginal rate of substitution**
   - The rate at which two goods can be traded off while keeping the subject indifferent

$$\begin{aligned} MRS(x_2, x_1) &= -\lim_{\Delta(x_1)\to 0} \frac{\Delta(x_2)}{\Delta(x_1)} \\ \text{such that } (x_1, x_2) &\sim (x_1 + \Delta(x_1), x_2 + \Delta(x_2)) \end{aligned}$$


Good 2
Bundle (x_1, x_2)
x_2
Δx_2
Bundle (y_1, y_2)
y_2
Δx_1
x_1
y_1
Good 1



- Question: Is MRS always well defined?

- If preferences can be represented by a utility function, then the equation of an indifference curve is given by

$$u(x) = \bar{u}$$

- Thus, if the utility function is differentiable we have

$$\sum_{i=1}^{N} \frac{\partial u(x)}{\partial x_i} dx_i = 0$$

- And so, in the case of two goods, the slope of the indifference curve is

$$\frac{dx_2}{dx_1} = -\frac{\frac{\partial u(x)}{\partial x_1}}{\frac{\partial u(x)}{\partial x_2}} = -MRS$$

which is another way of characterizing the MRS

Preferences over Commodity Bundles

- When thinking about preferences over commodity bundles it might be natural to assume that preferences have properties other than just
  - Completeness
  - Transitivity
  - Reflexivity
- Some of these we have come across before
  - (Strict) Monotonicity

$$\begin{aligned} x_n &\geq y_n \text{ for all } n \text{ and } x_n > y_n \text{ for some } n \\ \text{implies that } x &\succ y \end{aligned}$$

  - Monotonicity

$$\begin{aligned} x_n &\geq y_n \text{ for all } n \text{ implies } x \succeq y \\ x_n &> y_n \text{ for all } n \text{ implies } x \succ y \end{aligned}$$

  - Local Non-Satiation
- Examples?

- Another property often assumed is **convexity**

Definition

The preference relation $\succeq$ is convex if the upper contour set
$U_{\succeq}(x) = \{y \in X | y \succeq x\}$ is convex
i.e. for any x, z, y such that $y \succeq x$ and $z \succeq x$ and $\alpha \in (0, 1)$

$$(\alpha y + (1-\alpha)z) \succeq x$$

- What do convex indifference curves look like?
- Some alternative (equivalent) definitions of convexity
  - If $x \succeq y$ then for any $\alpha\ (0, 1)\ \alpha x + (1-\alpha)y \succeq y$
  - For all $w = \alpha x + (1-\alpha)y$ either $w \succeq x$ or $w \succeq y$

- What is convexity capturing?
  1. Mixtures are better than extremes
  2. If x is better than y, then going towards x is an improvement

Definition

A preference relation is **strictly convex** if x, z, y such that $y \succeq x$ and $z \succeq x$ and $\alpha \in (0, 1)$

$$(\alpha y + (1 - \alpha)z) \succ x$$

- Examples of preferences that are convex but not strictly so?

- What does the utility function of convex preferences look like?

Fact

A complete preference relation with a utility representation is convex if and only if it can be represented by a quasi concave utility function - i.e., for every x the set

$$\{y \in X | u(y) \geq u(x)\}$$

is convex

- Note that q-concave is weaker than concave
- It is **not** the case that continuous, convex preferences can necessarily be represented by concave utility functions
  - See homework

- A another property that preferences **can** have is **homotheticity**
  - The preference relation $\succeq$ is homothetic if $x \succeq y$ implies $\alpha x \succeq \alpha y$ for any $\alpha \geq 0$

Fact

A complete, strictly monotonic, continuous homothetic preference relation with a utility representation can be represented with a utility function which is homogenous of degree 1, i.e.

$$u(\alpha x_1, ... \alpha x_n) = \alpha u(x_1, ... x_n)$$

- What do homothetic indifference curves look like?
  - Recall that if u is homogenous of degree 1 then $\frac{\partial u(x)}{\partial \alpha x_1}$ is homogenous of degree 0 - i.e. $\frac{\partial u(x)}{\partial x_1} = \frac{\partial u(\alpha x)}{\partial x_1}$

$$\left.\frac{dx_2}{dx_1}\right|_x = -\frac{\frac{\partial u(x)}{\partial x_1}}{\frac{\partial u(x)}{\partial x_2}} = -\frac{\frac{\partial u(\alpha x)}{\partial x_1}}{\frac{\partial u(\alpha x)}{\partial x_2}} \left.\frac{dx_2}{dx_1}\right|_{\alpha x}$$

  - Slope of indifference curves remains the same along rays
- What is their economic intuition?
  - MRS depends only on the ratio of two goods
  - Going to nullify income effects in a handy way

- Finally, we might be interested in preferences that are **quasi linear**
  - The preference relation $\succeq$ is quasi linear in commodity 1 if $x \succeq y$ implies
$$(x + \varepsilon e_1) \succeq (y + \varepsilon e_1)$$
    for all $\varepsilon > 0$ and
$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Fact

A complete, continuous strictly monotonic, quasi linear preference relation with a utility representation can be represented with a utility function of the form

$$u(x) = v(x_2, ...x_k) + x_1$$

Quasi Linear Preferences

- What do quasi linear indifference curves look like?
  - MRS is unaffected by changes in q-linear good
  - Indifference curves are *parallel*
- What is their economic intuition?
  - Marginal utility of q-linear good is constant
  - q-linear good does not affect marginal utility of other goods
  - Often interpreted as 'wealth'

- We are now in the position to think about what the solution to the consumer's problem looks like
- We will think of the consumer's problem as defined by
  - A set of preferences $\succeq$
  - A set of prices $p \in \mathbb{R}^N_{++}$
  - A wealth level w
- With the problem being

$$\begin{aligned} \text{choose } x &\in \mathbb{R}^N_+ \\ \text{in order to maximize } &\succeq \\ \text{subject to } \sum_{i=1}^{N} p_i x_i &\leq w \end{aligned}$$

- **Question:** is the consumer's problem guaranteed to have a solution?
- Not without some further assumptions
- Here is a simple example
    - Let $N = 1$, $w = 1$ and $p_1 = 1$
    - Let preferences be such that higher numbers are preferred so long as they are less that 1, so

$$\begin{aligned} \text{If } x &< 1 \text{ then } x \succeq y \text{ iff } x \geq y \\ \text{If } x &\geq 1 \text{ the } x \succeq y \text{ iff } y \geq x \end{aligned}$$

- We need to add something else
- Any guesses what?

Theorem

If preferences $\succeq$ are continuous then the consumer's problem has a solution

- Proof follows fairly directly from Weierstrass Theorem!

Theorem

Any continuous function evaluated on a compact set has a maximum and a minimum

- Means that in order to guarantee existence we need three properties
  - Continuity of the function (comes from continuity of preferences)
  - Closedness of the budget set (comes from the fact that it is defined using weak inequalities)
  - Boundedness of the budget set (comes from the fact that we insist prices are strictly positive)

The Walrasian Demand Correspondence

- We are now in a position to define the Walrasian demand correspondence
- This is the amount of each good that the consumer will demand as a function of prices and income
- $x(p, w) \subset \mathbb{R}_+^N$ is the (set of) solution to the consumer's maximization problem when prices are p and wealth is w
  - i.e. the set of all bundles that maximize preferences (or equivalently utility) when prices are p and wealth is w
- Here are some straightforward properties of x when we maintain the assumptions of
  - Continuity
  - Local non-satiation

Fact

x is homogeneous of degree zero (i.e. $x(\alpha p, \alpha w) = x(p, w)$ for $\alpha > 0$)

- This follows from the fact that

$$\begin{aligned} & \left\{ x \in \mathbb{R}^n_+ | \sum_{i=1}^n p_i x_i \leq w \right\} \\ = & \left\{ x \in \mathbb{R}^n_+ | \sum_{i=1}^n \alpha p_i x_i \leq \alpha w \right\} \end{aligned}$$

Properties of the Demand Correspondence

Fact

Walras Law:

$$\sum_{i=1}^{n} p_i x_i = w$$

for any $x \in x(p, w)$

- This follows directly from local non-satiation

Rationalizing a Demand Correspondence

- We know that a demand correspondence must be homogeneous of degree zero and satisfy Walras Law
- Is **any** such function rationalizable?
  - i.e. there exists preferences that would give rise to that demand function as a result of optimization
- The answer is no, as the following example demonstrates
  - We will provide conditions that do guarantee rationalizability later in the course

Rationalizing a Demand Correspondence

- Example: Spending all one's money on the most expensive good:

$$x(p, w) = \begin{cases} (0, w/p_2) \text{ if } p_2 \geq p_1 \\ (w/p_1, 0) \text{ if } p_1 > p_2 \end{cases}$$

- This is homogenous of degree 0 and satisfies Walras law
- But cannot be rationalized (see diagram)

Properties of the Demand Correspondence

- Our final two properties are going to involve uniqueness and continuity of x
- Further down the road it will be very convenient for
  - x to be a function (not a correspondence)
  - x to be continuous
- What can we assume to guarantee this?

Properties of the Demand Correspondence

- First: do we have uniqueness?
- No! (see diagram)
- Here, convexity will come to our rescue

Fact

If $\succeq$ is convex then $x(p, w)$ is a convex set. If $\succeq$ is strictly convex then $x(p, w)$ is a function

- Proof comes pretty much directly from the definition and the fact that the budget set is convex

Properties of the Demand Correspondence

- In fact, if x is a function then we also get continuity

Fact

If x is single values and $\succeq$ is continuous then x is continuous

- Proof comes directly from the theorem of the maximum

Properties of the Demand Correspondence

Theorem (The Theorem of the Maximum)

Let

- X *and* Y *be metric spaces (*Y *will be the set of things that can be chosen,* X *the set of parameters)*
- $\Gamma : X \rightrightarrows Y$ *be compact valued and continuous (this is the budget set)*
- $f : X \times Y \to \mathbb{R}$ *be continuous, (this is the utility function)*
  Now define $y^* : X \rightrightarrows Y$ *as the set of maximizers of* f *given parameters* x
  $$y^*(x) = \arg\max_{y \in \Gamma(x)} f(x, y)$$
  and define $f^* : X \rightrightarrows Y$ *as the maximized value of* f *for* f *given parameters* x
  $$f^*(x) = \max_{y \in \Gamma(x)} f(x, y)$$

Properties of the Demand Correspondence

Theorem (The Theorem of the Maximum)

Then

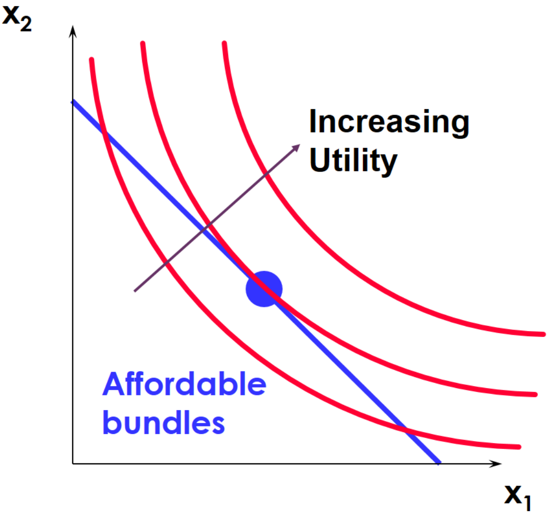
1. y^* *is upper hemi-continuous and compact valued*
   - *i.e for* $x_n \longrightarrow x$ *and* $y_n \in y^*(x_n)$ *such that* $y_n \longrightarrow y$ *implies* $y \in y^*(x)$
2. f^* *is continuous*

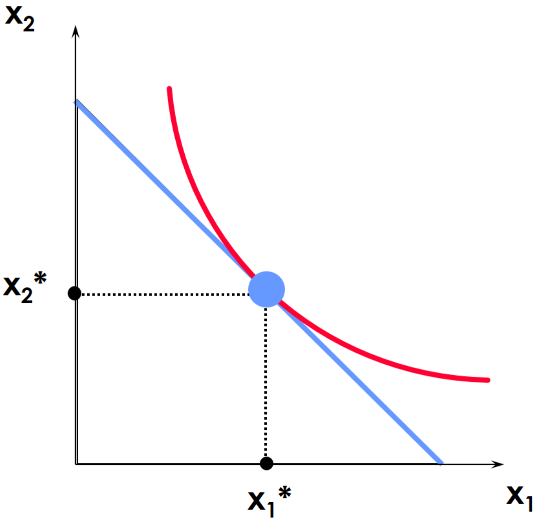
Corollary

If y^* *is single valued it is continuous*

Tangency Conditions

- Graphically, what does the solutions to the consumer's problem look like?
- Here it is useful to think in two dimensions


x2
Increasing
Utility
Affordable
bundles
x1

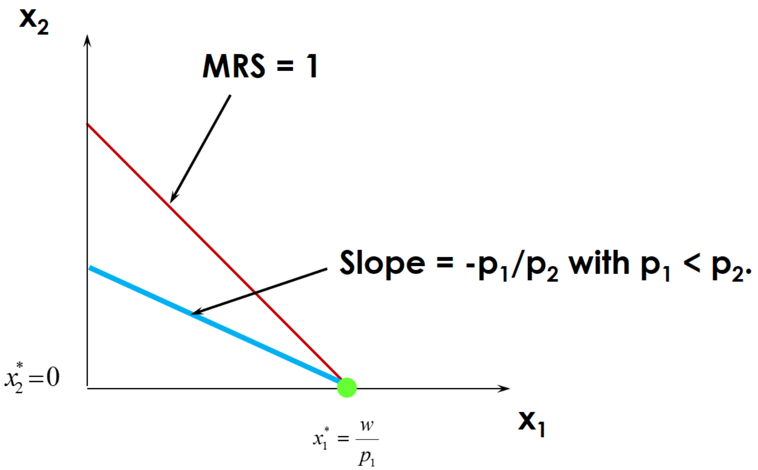

x2
x2*
x1*
x1

Tangency Conditions

- If the solution to the consumer's problem is interior, then
  - The indifference curve
  - The budget line

  are tangent to each other
- Implies that the marginal rate of substitution is the same as the price ratio
- This makes intuitive sense
  - The rate at which goods can be traded off against each other in the market
  - is equal to the rate at which they can be traded off leaving the consumer indifferent
  - If not, then utility could be increased by switching to the 'cheaper' good

- What about corner solutions?
- For example, none of good 2 is purchased
- Here, the indifference curve and the price line need not be equal
- But the price line must be **shallower** than the slope of indifference curve


x_2
MRS = 1
Slope = $-p_1/p_2$ with $p_1 < p_2$.
$x_2^* = 0$
x_1
$x_1^* = \frac{w}{p_1}$

- In the case in which utility is continuously differentiable, we can use the **Kuhn Tucker** (necessary) conditions to capture this intuition
- For the problem

$$\max u(x)$$

$$\begin{aligned} \text{subject to } \sum_{i=1}^{n} p_i x_i - w &= 0 \\ -x_i &\leq 0 \ \forall i \end{aligned}$$

- We can set up the Lagrangian for the problem

$$u(x) - \lambda \left(\sum_{i=1}^{n} p_i x_i - w \right) - \sum_{i=1}^{n} \mu_i \left(-x_i \right)$$

Kuhn Tucker Conditions

- A necessary condition of a solution to the optimization problem x^* is the existence of λ, and $\mu_i \geq 0$ such that

$$\begin{aligned} \frac{\partial u(x^*)}{\partial x_i} - \lambda p_i + \mu_i &= 0 \\ \text{and } x_i^* . \mu_i &= 0 \text{ for all } i \end{aligned}$$

- So, if $x_i^* > 0$ then $\mu_i = 0$ and

$$\frac{\partial u(x^*)}{\partial x_i} = \lambda p_i$$

- If $x_i^* = 0$ then $\mu_i \geq 0$ and so

$$\frac{\partial u(x^*)}{\partial x_i} \leq \lambda p_i$$

Kuhn Tucker Conditions

- This can be summarized compactly by saying, that for a solution x^*

$$\begin{aligned} \nabla u(x^*) &\leq \lambda p \\ x^* \left[\nabla u(x^*) - \lambda p\right] &= 0 \end{aligned}$$

- Note that this implies that

$$\frac{\frac{\partial u(x^*)}{\partial x_i}}{\frac{\partial u(x^*)}{\partial x_j}} = \frac{p_i}{p_j}$$

- if x_i and x_j are both strictly positive

- These conditions are **necessary** for an optimum
- They become sufficient if preferences are convex
- This follows from the KT theorem, but Rubinstein provides a nice direct proof

Theorem

If $\succeq$ are strongly monotonic, convex, continuous and differentiable*, and

1. $px^* = w$
2. *for every k such that $x_k^* > 0$*

$$\frac{\frac{\partial u(x^*)}{\partial x_k}}{p_k} \geq \frac{\frac{\partial u(x^*)}{\partial x_j}}{p_j}$$

Then x^* is a solution to the consumer's problem

- What do we mean by differentiable*?
- Not going to go into this formally
- The important part for us is that it means the following.
- Define d as an 'improving direction' at x if there exists a λ^* such that, for all $0 < \lambda \leq \lambda^*$

$$x + \lambda d \succ x$$

- If $\succeq$ is differentiable then d is an improving direction iff $d.\nabla u > 0$

The Demand Function and Prices

- Notice that so far we have **not** said that demand must decrease in price
- This is because it is not generally true!
- Example:

$$u(x_1, x_2) = \begin{cases} x_1 + x_2 \text{ if } x_1 + x_2 < 1 \\ x_1 + 4x_2 \text{ if } x_1 + x_2 \geq 1 \end{cases}$$

- Consider $x((p_1, 2), 1)$
  - What happens in the range $p_1 \in [1, \frac{1}{2}]$
  - Maximize utility by spending everything on good 2 while making sure $x_1 + x_2 \geq 1$

$$x((p_1, 2, 1) = (1/(2 - p_1), (1 - p_1)/(2 - p_1))$$

- As we shall see in more detail later, this is basically because change in prices changes **income** as well as relative prices
- This points to a version of the above statement which is true

Theorem

Let x be a rationalizable demand function that satisfies Walras' law and $I' = p'x(p, I)$. Then

$$[p' - p][x(p', I') - x(p.I)] \leq 0$$

- Means that if price of good i falls then demand for it cannot also fall
- Note that this is slightly different to the claim in Rubinstein