Consumer Choice 1 - Proofs

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- Another property often assumed is **convexity**
  - The preference relation $\succeq$ is convex if the upper contour set $U_{\succeq}(x) = \{y \in X | y \succeq x\}$ is convex
  - i.e. for any x, z, y such that $y \succeq x$ and $z \succeq x$ and $\alpha \in (0, 1)$

$$(\alpha y + (1 - \alpha)z) \succeq x$$

- What do convex indifference curves look like?
- Some alternative (equivalent) definitions of convexity
  - If $x \succeq y$ then for any $\alpha\ (0, 1)\ \alpha x + (1 - \alpha)y \succeq y$
  - For all $z = \alpha x + (1 - \alpha)y$ either $z \succeq x$ or $z \succeq y$

- Proof that all are equivalent
  - $1 \Rightarrow 2$: Direct: $x \succeq y$ and $y \succeq y$ so $\alpha x + (1-\alpha)y \succeq y$
  - $2 \Rightarrow 3$: Either $x \succeq y$ in which case $z \succeq y$ or $y \succeq x$ and so $z \succeq x$
  - $3 \Rightarrow 1$: WLOG assume $y \succeq z$, then $\alpha y + (1-\alpha)z \succeq z \succeq x$

Fact

A complete preference relation with a utility representation is convex if and only if it can be represented by a quasi concave utility function - i.e., for every x the set

$$\{y \in X | u(y) \geq u(x)\}$$

is convex

Proof.

Immediate let $y \succeq x$ and $z \succeq x$ then $u(y) \geq u(x)$ and $u(z) \geq u(x)$, by q concavity $u(\alpha y + (1-\alpha)z) \geq u(x)$ and so $\alpha y + (1-\alpha)z \succeq x$
Similarly, say $u(y) \geq u(x)$ and $u(z) \geq u(x)$ implies $y \succeq x$ and $z \succeq x$ by convexity $\alpha y + (1-\alpha)z \succeq x$ and so $u(\alpha y + (1-\alpha)z) \geq u(x)$ □

Fact

A complete, increasing, continuous homothetic preference relation with a utility representation can be represented with a utility function which is homogenous of degree 1, i.e.

$$u(\alpha x_1, ...\alpha x_n) = \alpha u(x_1, ...x_n)$$

Proof.

You have shown for homework that a utility representation for such preferences is given by

$$u(x) = a | \{a, a, ...a\} \sim x$$

If $\succeq$ *is homothetic, then*

$$\begin{aligned} \{a, a, ...a\} &\sim x \rightarrow \{\alpha a, \alpha a, ...\alpha a\} \sim \alpha x \\ &\Rightarrow u(\alpha x) = \alpha a = \alpha u(x) \end{aligned}$$

Fact

A complete, continuous strictly monotonic, quasi linear preference relation with a utility representation can be represented with a utility function of the form

$$u(x) = v(x_2, ...x_k) + x_1$$

Lemma

Under the conditions of the proof, for any $x_2, ...x_k$ *there exists a* $v(x_2, ...x_k)$ *such that*

$$(0, x_2, ...x_k) \sim (v(x_2, ...x_k), 0, ..., 0)$$

Lemma

Proof.

(for k=2): We want to want to show that for every x_2 there is some x_1 such that $(x_1, 0) \sim (0, x_2)$. Thus we need to show that the following set is empty

$$T = \{t | (0, t) \succ (x_1, 0) \ \forall \ x_1\}$$

Assume not □

Proof.

[Proof (cont)] First assume $m \in T$. Then $m > 0$ and $(1, m) \succ (0, m)$ (MON) and an ε such that $(1, m - \varepsilon) \succ (0, m)$ (CONT) and so $(1, m - \varepsilon) \succ (x_1 + 1, 0)\ \forall\ x_1$ (as $M \in T$). As $m = \inf T$, there exists an x_1^* such that $(x_1^*, 0) \succeq (0, m - \varepsilon)$, but by quasi linearity $(x_1^* + 1, 0) \succeq (1, m - \varepsilon)$ a contradiction

Next assume that $m \notin T$. Then $(x_1^*, 0) \sim (0, m)$ for some x_1^*, and by monotonicity $(x_1^* + 1, 0) \succ (0, m)$. By continuity there exists an $\varepsilon > 0$ such that $(x_1^* + 1, 0) \succ (0, x_2)$ for any $m + \varepsilon > x_2 > m$ meaning that such x_2 are not in T and so $m \neq \inf T$

Thus, by the standard argument, we can set $v(x_2) = \inf(x_1 | (x_1, 0) \succeq (0, x_2))$. We have just show that this set is non-empty, and by continuity we know that this implies that $(v(x_2), 0) \sim (0, x_2)$ □

Proof.
[Proof (of Claim)] We want to show that $x_1 + v(x_2, ...x_k)$ represents $\succeq$. Note that for any x

$$\begin{aligned} (0, x_1, ...x_k) &\sim (v(x_2, ..., x_k), 0, ...0) \Rightarrow \\ (x_1 + v(x_2, ..., x_k), 0...0) &\sim x \text{ (by q linearity)} \end{aligned}$$

Thus by strong monotonicity we are done

$$\begin{aligned} x &\succeq y \\ &\Rightarrow (x_1 + v(x_2, ..., x_k), 0...0) \succeq (y_1 + v(y_2, ..., y_k), 0...0) \end{aligned}$$

□

Theorem

If $\succeq$ are strongly monotonic, convex, continuous and differentiable*, and

1. $px^* = w$
2. *for every k such that $x_k^* > 0$*

$$\frac{\frac{\partial u(x^*)}{\partial x_k}}{p_k} \geq \frac{\frac{\partial u(x^*)}{\partial x_j}}{p_j}$$

Then x^* is a solution to the consumer's problem

Proof.

Assume not, then there exists a y such that $py \leq px^*$ but $y \succ x^*$

Let $\mu = \frac{\frac{\partial u(x^*)}{\partial x_k}}{p_k}$ for all k st $x_k^* > 0$ and note

$$\begin{aligned} 0 &\geq p(y - x^*) \\ &= \sum_k p^k (y_k - x_k^*) \\ &\geq \sum_k \frac{\partial u(x^*)}{\partial x_k} \frac{(y_k - x_k^*)}{\mu} \end{aligned}$$

This follows from the fact that if $x_k^* > 0$ then $p^k = \frac{\partial u(x^*)}{\partial x_k} / \mu$, and if not $(y_k - x_k^*) \geq 0$ and $p^k \geq \frac{\partial u(x^*)}{\partial x_k} / \mu$

Thus

$$0 \geq \nabla u(x^*)(y - x^*)$$

□

Proof.

But the fact that $y \succ x^*$, along with strong monotonicity, continuity, and convexity means that

$$\lambda x^* + (1-\lambda) y \succ x^*$$

(see Rubinstein)

Thus $(y - x^*)$ is an improving direction - a contradiction □

Theorem

Let x be a rationalizable demand function that satisfies Walras' law and $I' = p'x(p, I)$. Then

$$[p' - p][x(p', I') - x(p.I)] \leq 0$$

Proof.

$$\begin{aligned} & [p' - p][x(p', I') - x(p.I)] \\ = \; & p'x(p', I') - p'x(p.I) - px(p', I') + px(p.I) \\ = \; & I' - I + px(p.I) - px(p', I') \end{aligned}$$

If $px(p.I) - px(p', I') > 0$ then $x(p.I)$ strictly preferred to $x(p', I')$
But as $p'x(p.I) = I'$ $x(p', I')$ weakly preferred to $x(p.I)$ □

- **Property 3:** v is quasiconvex: i.e. the set

$$\{(p, w) | v(p, w) \leq \bar{v}\}$$

is convex for all $\bar{v}$

Proof.

Take p, w and p', w' in the set and let $p'' = \alpha p + (1 - \alpha) p'$ and $w'' = \alpha w + (1 - \alpha) w'$.

NTS that $v(p'', w'') \leq \bar{v}$. Assume not. Then there exists an x st $u(x) > \bar{v}$ and

$$\begin{aligned} p'' x &\leq w'' \Rightarrow \\ \alpha p x + (1 - \alpha) p' x &\leq \alpha w + (1 - \alpha) w' \Rightarrow \\ \alpha (p x - w) + (1 - \alpha)(p' x - w') &\leq 0 \end{aligned}$$

□

Proof.

Contradiction □

Theorem

If u is monotonic and continuous then if x^* is a solution to the prime problem with prices p and wealth w it is a solution to the dual problem with prices p and utility $v(p, w)$

Proof.

- Assume not, then there exists a bundle $\bar{x}$ such that

$$u(\bar{x}) \geq v(p, w) = u(x^*)$$

with

$$\sum p_i \bar{x}_i < \sum p_i x_i^* = w$$

- But this means, by monotonicity, that there exists an $\varepsilon > 0$ such that, for

$$x' = \begin{pmatrix} \bar{x}_1 + \varepsilon \\ \bar{x}_2 + \varepsilon \\ \vdots \\ \bar{x}_N + \varepsilon \end{pmatrix}$$

$$\sum p_i x_i' < w$$

□

Proof.

- By monotonicity, we know that $u(x') > u(\bar{x}) \geq u(x^*)$, and so x^* is not a solution to the prime problem

□

Theorem

If u is monotonic and continuous then if x^* is a solution to the dual problem with prices p and utility u^* it is a solution to the prime problem with prices p and wealth $\sum p_i x_i^*$

Proof.

- Assume not, then there exists a bundle $\bar{x}$ such that

$$\sum p_i \bar{x}_i \leq \sum p_i x_i^*$$

with

$$u(\bar{x}) > u(x^*) \geq u^*$$

- By continuity, there exists an $\varepsilon > 0$ such that, for all $x' \in B(\bar{x}, \varepsilon)$, $u(x') > u(x^*)$

□

Proof.

- In particular, there is an $\varepsilon > 0$ such that

$$x' = \begin{pmatrix} \bar{x}_1 - \varepsilon \\ \bar{x}_2 - \varepsilon \\ \vdots \\ \bar{x}_N - \varepsilon \end{pmatrix}$$

  and $u(x') > u(x^*) \geq u^*$

- But $\sum p_i x_i' < \sum p_i \bar{x}_i \leq \sum p_i x_i^*$, so x^* is not a solution to the prime problem.

□

Properties of the Hicksian Demand Function

- **Fact 3**: If preferences are convex then h is a convex set. If preferences are strictly convex then h is unique
  - Proof: say that x and y are both in $h(p,u)$. Then

$$\sum p_i x_i = \sum p_i y_i = e(p,u)$$

  - Implies that for any $\alpha \in (0,1)$ and $z = \alpha x + (1-\alpha)y$

$$\begin{aligned} \sum p_i z_i &= \sum p_i \left(\alpha x_i + (1-\alpha)y_i\right) \\ &= \alpha \sum p_i x_i + (1-\alpha) \sum p_i y_i \\ &= e(p,u) \end{aligned}$$

  - Also, as preferences are convex, $z \succeq x$, and so $u(z) \geq u(x) = u$
  - If preferences are **strictly** convex, then $z \succ x$
  - But, by continuity, exists $\varepsilon > 0$ such that $z' \succ x$ all $z' \in B(x, \varepsilon)$
  - Implies that there is a z' such that $u(z') > u$ and $\sum p_i z_i < \sum p_i x_i$

Properties of the Expenditure Function

- **Fact 4:** e is concave in p
  - Proof: fix a $\bar{u}$. we need to show that

$$e(p'', \bar{u}) \geq \alpha e(p, \bar{u}) + (1-\alpha)e(p', \bar{u})$$

where

$$p'' = \alpha p + (1-\alpha)p'$$

  - Let $x'' \in h(p'', \bar{u})$, then

$$\begin{aligned} e(p'', \bar{u}) &= \sum p_i'' x_i'' \\ &= \sum \left(\alpha p_i + (1-\alpha)p_i'\right) x_i'' \\ &= \alpha \sum p_i x_i'' + (1-\alpha) \sum p_i' x_i'' \\ &\geq \alpha \sum p_i x_i + (1-\alpha) \sum p_i' x_i' \\ &= \alpha e(p, \bar{u}) + (1-\alpha) e'(p, \bar{u}) \end{aligned}$$

where $x \in h(p, \bar{u})$ and $x' \in h(p', \bar{u})$