Production

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GR6211 - Microeconomic Analysis 1

Introduction

- Up until now we have dealt exclusively with one type of economic agent: the consumer
  - Defined by a set of preferences
- We are now going to deal (very quickly) with the other basic type of economic agent: the firm
- As much as possible I am going to try to convince you that you already know how to deal with the problem of a firm
  - Familiar problems with a new name

What is a Firm?

- The reason we really need firms for economic analysis is that so far we are missing something important
  - Where does stuff come from for consumers to consume?
- We could assume that it is just lying around
  - As in an endowment economy
- But this misses out the fact that there are lots of economic actors that produce the stuff that consumers buy
  - i.e. firms
- Moreover, these firms don't just sell stuff that they were endowed with
- They **convert** inputs into outputs
  - e.g. a baker converts labor and flour into bread
  - a university converts students and professors into knowledge (?)

- So, while in principle a firm could be characterized by lots of things
  - Its organization
  - Its motivation
  - Who owns it
- We will define it by its ability to transform things from one type to another
  - i.e. its technology.

The Production Vector

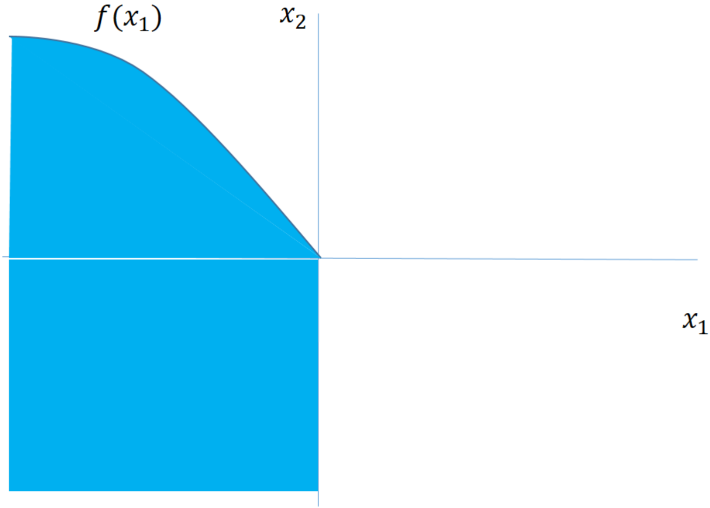
- Let's imagine we are in an L commodity world
- A production vector is just an L length vector which describes the net output of each good
    - e.g.

$$\begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$$

- Would mean **using** 3 units of good 1 to produce 2 units of good 2 and 1 unit of good 3
- A firm is simply defined by its production set $Y \subset \mathbb{R}^L$
- This is the set of **feasible** production vectors for the firm

The Production Set

- A simple example with $L = 2$
- The firm uses commodity 1 to produce commodity 2
  - i.e. x_1 is always (weakly) negative
  - Call good one ‘labor’ and good 2 ‘sausages’
- The maximal amount of good 2 that can be produced given consumption of good 1 is given by $f(x_1)$
- But the firm can always 'throw away' good 2
- Then the production set looks like this....



$f(x_1)$
x_2
x_1

Inputs and Outputs

- Often we will assume that for a firm commodities are split into **inputs** and **outputs**
  - Outputs are produced in positive amounts $(q_1, ...q_M)$
  - Inputs are used in positive amounts $(z_1, ...z_{L-M})$
- One special case is the one in which there is only 1 output
  - This is essentially the only case that we will deal with
- In this case we can define the production set using the **production function**

$$f(z_1, ..z_N)$$

- This is the maximal amount of q that can be produced with inputs $z_1, ...z_N$
  - When dealing with production functions it will be more convenient to treat the z's as positive numbers
  - so

$$f(z_1, ..z_N) = \max\{x|(x, -z_1, .. - z_N) \in Y\}$$

Properties of the Production Set

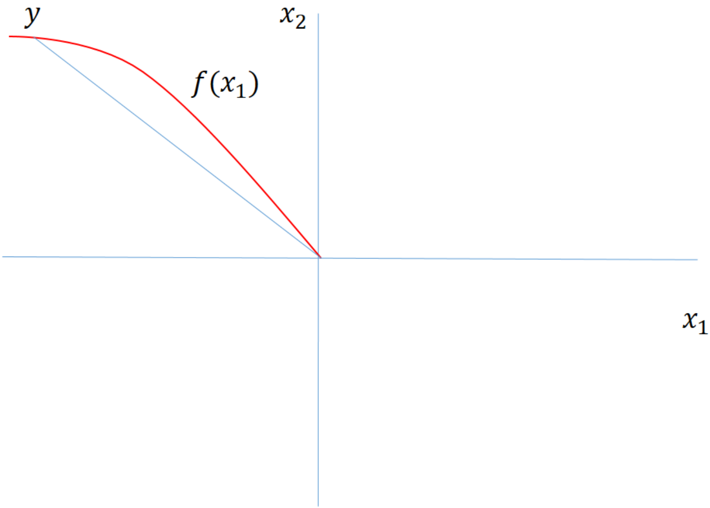
- Here are some properties that we might want the production set to have

1. Y **is nonempty and closed**. This will help ensure that there is (a) something to study and (b) optimization problems involving the firm will have a solution
2. $Y \cap \mathbb{R}^L_+ = 0$. This says (a) that the firm can do nothing and (b) that there is no free lunch
3. **Free disposal - i.e. if** $y \in Y$ and $y' \leq y$ then $y' \in Y$. Firms can always dispose of any commodity at zero cost.

Returns to Scale

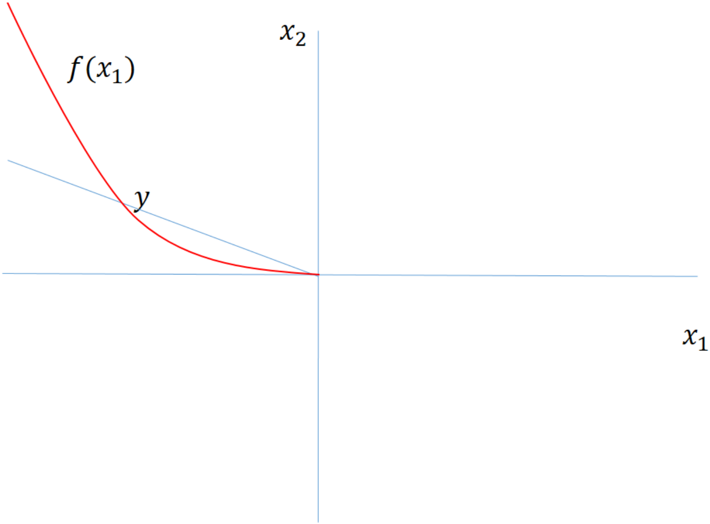
- A crucial characteristic of the production function is its **returns to scale**
  - Broadly speaking, does a firm become more or less efficient as it produces more output

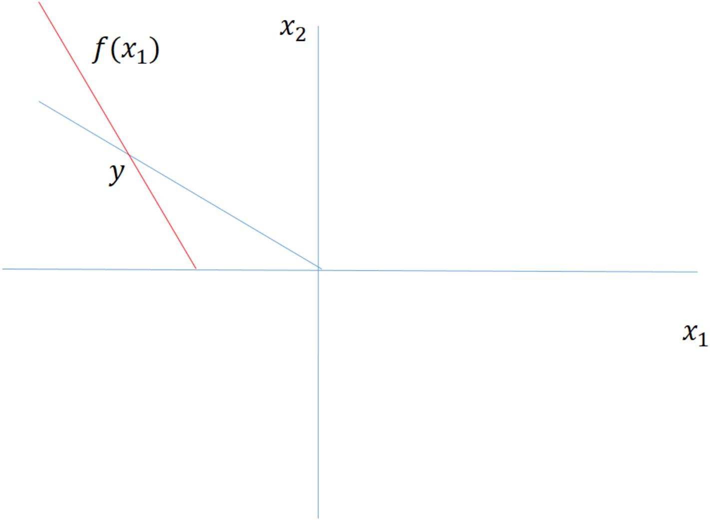
1. **Non-increasing returns to scale:** if $y \in Y$ then $\alpha y \in Y$ for $\alpha \in [0, 1]$
2. **Non-decreasing returns to scale:** if $y \in Y$ then $\alpha y \in Y$ for $\alpha \geq 1$
3. **Constant returns to scale:** if $y \in Y$ then $\alpha y \in Y$ for $\alpha \geq 0$



y
x_2
$f(x_1)$
x_1

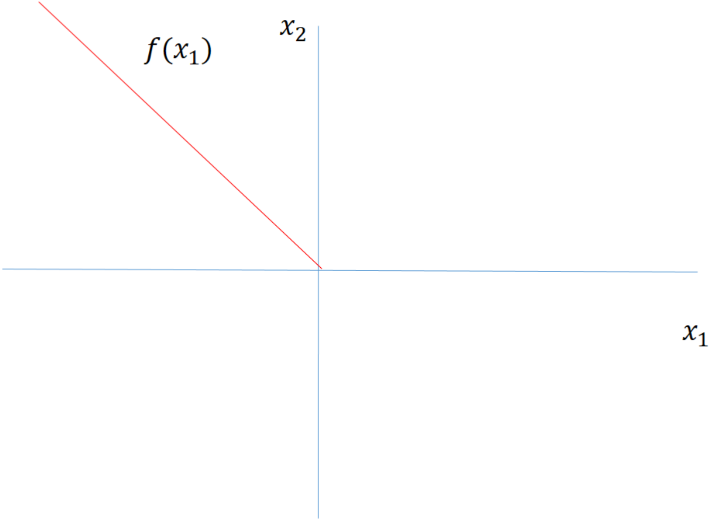
Non-Decreasing

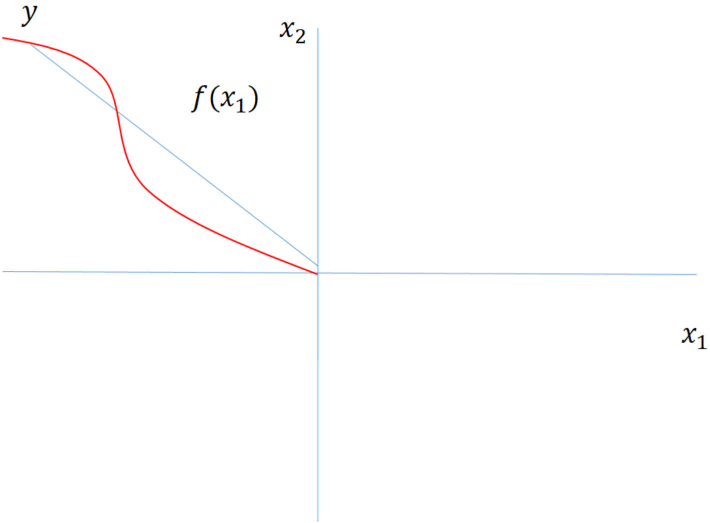




x_2
$f(x_1)$
y
x_1

Constant





y
x_2
$f(x_1)$
x_1

Convexity

- One assumption that is very handy (as usual) is convexity
- What type of returns to scale are necessary for Y to be convex?
  - Non-increasing!
- Is this sufficient for Y to be convex?
- Not if there are multiple inputs
- Also requires that mixtures of inputs are more efficient than extremes
  - Analogous with consumption

- So far we have used Y to define a firm in the same way that we used preferences to define an consumer
- But we have not defined an optimization problem for the firm!
- In the end, we will want firms to maximize profits
- But before that it is going to be extremely useful to define the **cost minimization problem**

Definition

Consider a firm which produces 1 output using L inputs. Let $w \in \mathbb{R}^L_{++}$ be the vector of input prices. The cost minimization problem is

$$\min_{z \geq 0} z.w$$
$$\text{subject to } f(z) \geq q$$

Let $c(q, w)$ be the cost function and $z(q, w)$ be the factor demand correspondence

Some Things to Note

- Note

1. We are assuming that the firm is a **price taker** - i.e. it treats the prices of inputs as fixed
2. You have seen this problem before!
    - It is the expenditure minimization problem!
    - If fact, often we use the same functional forms for technology and preferences, such as Cobb Douglas
    - However, in this case the problem makes a lot more sense, and the cost function is easier to interpret than the expenditure function

The Case of 1 Input

- Let's start off with the case of 1 input
- In this case the cost minimization problem is easy!
- For any q, we have

$$\begin{aligned} z(q, w) &= f^{-1}(q) \\ c(q, w) &= w f^{-1}(q) \end{aligned}$$

- That was boring!
  - Note that either we are assuming that f is strictly monotonic, or z may be a correspondence

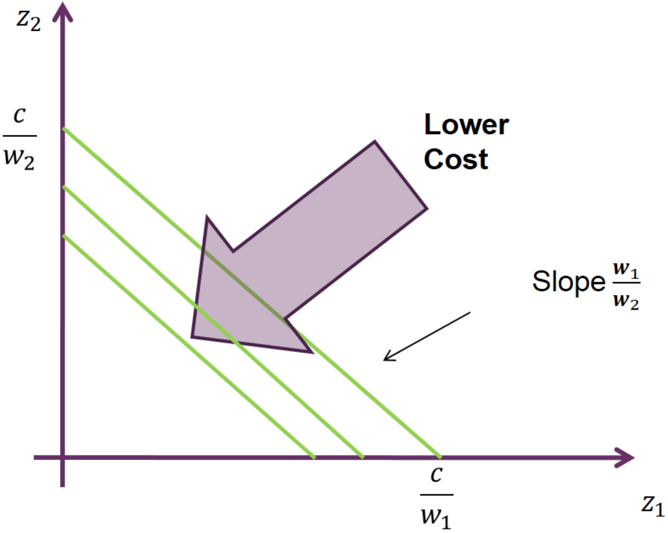
- However, we can still learn something about the relationship between the cost function and the production function
- Define marginal costs as the derivative of the cost function with respect to q
- As

$$\begin{aligned} c(q,w) &= wf^{-1}(q) \\ &\Rightarrow \frac{\partial c(q,w)}{\partial q} = \frac{w}{\frac{\partial f(z)}{\partial z}\big|_{z=z(q,w)}} \end{aligned}$$

- So, if $\frac{\partial f(z)}{\partial z}$ is **increasing** then $\frac{\partial c(q,w)}{\partial q}$ is **decreasing**
- So if f is concave (i.e. returns to scale are decreasing) then c is convex (i.e. marginal costs are increasing)
- if f is convex then c is concave
- If f is linear then so is c

The Case of Multiple Inputs

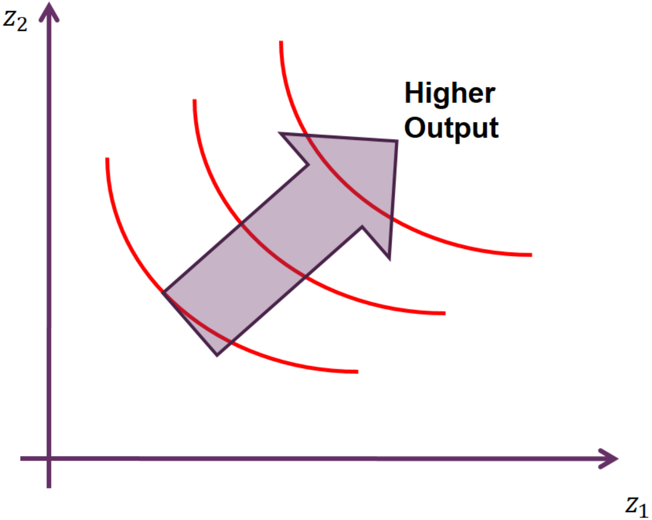
- The case of multiple inputs is more interesting!
- Now there are multiple different collection of inputs that will generate the same output
- Have to choose the cheapest one
- Let's start with some pictures in 2 dimensions
- First, we need iso-cost lines
  - This is what we are trying to minimize



z_2
$\frac{c}{w_2}$
Lower
Cost
Slope $\frac{w_1}{w_2}$
$\frac{c}{w_1}$
z_1

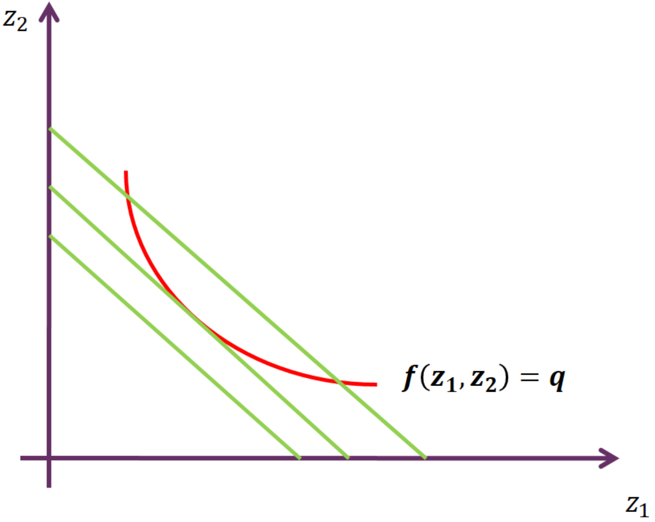
Iso-Output Lines

- Next we need the constraint
- i.e. the set
$$\{z|f(z) \geq q\}$$
- Clearly the shape of this is going to depend on the production function
  - Assume that f is weakly monotonic
  - If Y is convex then the iso output lines will be convex


z_2
Higher
Output
z_1

Solution

- So we know what a solution is going to look like.....



z_2
$f(z_1, z_2) = q$
z_1

- Assuming an interior solution it means that the slope of the iso-cost line is the same as that of the iso-output line
- Slope of the iso cost line is

$$\frac{w_1}{w_2}$$

- Iso-output line defined by

$$\begin{aligned} f(z_1, z_2) &= q \\ &\Rightarrow \frac{\partial f}{\partial z_1} dz_1 + \frac{\partial f}{\partial z_2} dz_2 = 0 \\ &\Rightarrow \frac{dz_2}{dz_1} = -\frac{\frac{\partial f}{\partial z_1}}{\frac{\partial f}{\partial z_2}} \end{aligned}$$

- This is the **marginal rate of technical substitution between** z_1 and z_2
  - The rate at which the firm can trade off z_1 and z_2 keeping output constant

Solution

- As with the consumer's problem we can set up the Lagrangian

$$\sum_L z_l w_l - \lambda(f(z) - q) - \sum_L \mu_l(-z_l)$$

- And get the solution

$$w_l \leq \lambda \frac{\partial f}{\partial z_l} \text{ and } w_l = \lambda \frac{\partial f}{\partial z_l} \text{ if } z_l > 0$$

Solution

- Here the Lagrangian really approach really comes into its own
- What is λ?
- Recall the envelope theorem

$$\frac{\partial c(w,q)}{\partial q} = \frac{\partial f(z(w,\bar{q}) : w,q)}{\partial q} - \lambda \frac{\partial g_n(z(w,q) : w,q)}{\partial q}$$

- Recall that $\frac{\partial g_n(z(w,q):w,q)}{\partial q}$ is the change in the value of the constraint with respect to q
  - In this case it is equal to 1
- $\frac{\partial f(z(w,q):w,q)}{\partial q}$ is the direct impact of a change in q on the objective function
  - In this case equal to zero

Solution

- So

$$\frac{\partial c(w, \bar{q})}{\partial q} = \lambda$$

- It is the change in the object function with respect to a change in the constraint
- i.e. it is how $c(q, w)$ changes with q
- i.e. it is the marginal cost!
- Rather than solving for the function c and then differentiating with respect to q we can take the marginal cost straight from the Lagrangian!

Properties of Demands and Costs

- Now we will list some properties of z and c
- I'm not going to provide proofs
  - Easy adaptation of the proofs from consumption case
- Assume throughout that Y is closed and satisfies the free disposal property

Properties of Demands and Costs

- **Property 1:** z is a homogenous of degree 0 in w
- **Property 2:** c is homogenous of degree 1 in w and nondecreasing in q
- **Property 3:** c is a concave function of w
- **Property 4:** If the set $\{z|f(z) \geq q\}$ is convex the $z(w, q)$ is a convex set. If it is a strictly convex set then $z(w, q)$ is unique

- **Property 5:** (Shephard's lemma) If $z(w, q)$ is unique then $c(w, q)$ is differentiable with respect to w and

$$\frac{\partial c(w, q)}{\partial w_l} = z_l(w, q)$$

- **Property 6:** If f is homogeneous of degree 1 (i.e. constant returns to scale) then c and z are homogenous of degree 1 in q
- **Property 7:** If f is concave then c is convex in q

Profit Maximization

- Now we are in a position to define the ultimate goal of the firm: **prrrrrrrrrrofit!**
  - Not the only possible assumption, but a very standard one
- We will assume that the firm is a price taker on the output side as well
  - Output can be sold at a constant price p
- This is a ‘perfect competition’ assumption
- You will come across other alternatives later in the course
  - Monopoly
  - Oligopoly

Definition

Consider a firm which produces 1 output using L inputs. Let $w \in \mathbb{R}^L_{++}$ be the vector of input prices and p' be the output price. Let $p = (p', w)$ be the vector of all prices The profit maximization problem is

$$\max_{z \geq 0} p' f(z) - w.z$$

with $\pi(p)$ being the associated profit function and $y(p)$ the set of vectors in Y that maximize profit

- What do the first order conditions look like?

$$\frac{\partial f}{\partial z_l} \leq \frac{w_l}{p'}, \text{ with } \frac{\partial f}{\partial z_l} = \frac{w_l}{p'} \text{ if } z_l > 0$$

- Marginal product of an input z_l is equal to its price (in terms of output)
- Note that if Y is convex, these first order conditions are also sufficient

Properties of Profit Functions and Supply Correspondences

- **Property 1:** y is a homogenous of degree 0
- **Property 2:** π is homogenous of degree 1
- **Property 3:** π is a convex function
- **Property 4:** If Y is convex then y is convex. If Y is strictly convex then y is unique
- **Property 5:** (Hotelling's Lemma) If $y(p)$ is unique, then π is differentiable at p and

$$\frac{\partial \pi(p)}{\partial p'} = y(p)$$

Properties of Profit Functions and Supply Correspondences

- What about comparative statics?
- What happens to the supply of outputs and demand for inputs as their own prices change
- It turns out that they are well behaved
  - Output is (weakly) increasing in output prices
  - Input demand is (weakly) decreasing in its own price
- This is because we are basically solving for 'compensated' demand functions, so the law of compensated demand holds

- It should be fairly easy to see that profit maximization problem is the same as

$$\max_{q \geq 0} p'q - c(w, q)$$

- And choosing $z \in z(w, q)$
- What are the first order conditions here?

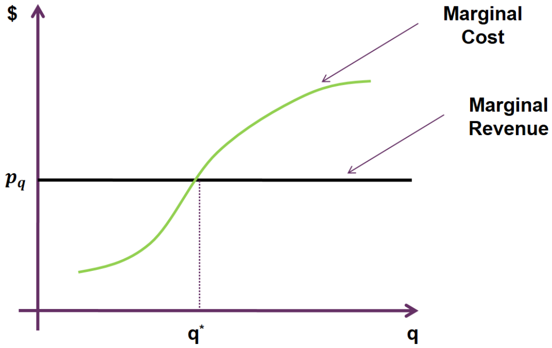
$$p' = c'(q)$$

$$\text{marginal revenue} = \text{marginal cost}$$

- What about second order conditions?
- Requires

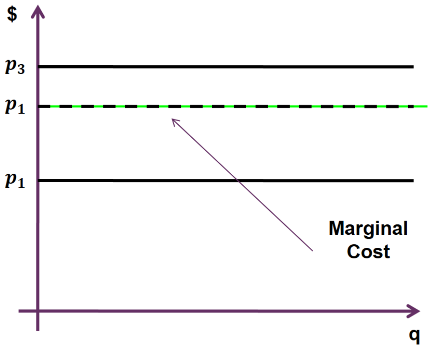
$$c''(q) \geq 0$$

  - i.e. marginal costs to be increasing
  - i.e. decreasing returns to scale
- This defines three basic cases

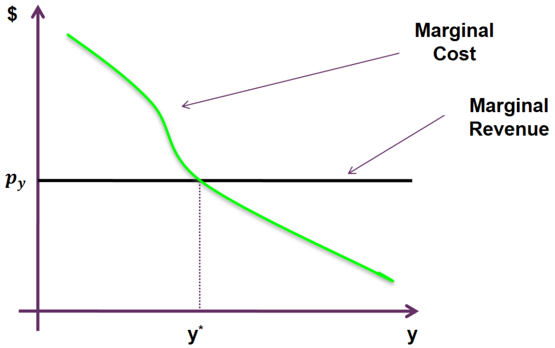


- (Typically) Interior solution
- Supply upward sloping
- Average cost below marginal cost

Constant Returns to Scale

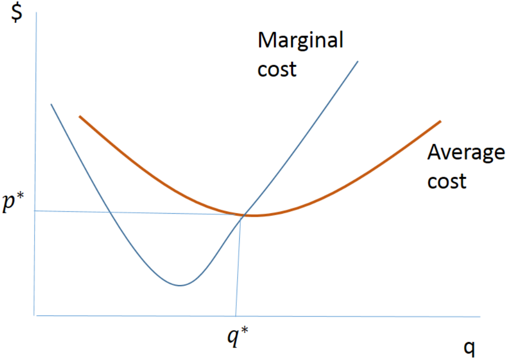


- (Typically) corner solution
- Supply a step function (zero below marginal cost, infinite at marginal cost)
- Average cost equal to marginal cost



- Corner solution
- Supply a step function (zero if price below the lowest possible marginal cost, infinite otherwise)
- Average cost higher than marginal cost

- A popular textbook example is a case when returns to scale are initially increasing then decreasing



- Note that
  - The marginal cost curve crosses the average cost curve at its nadir
  - For any $p < p^*$ at any level of output, average cost less than price

$$\begin{aligned} &\Rightarrow \quad p - AC < 0 \\ &\Rightarrow \quad pq - c(q) = \pi < 0 \end{aligned}$$

  - so firm will produce 0
  - For $p > p^*$ there are potentially two points at which $c'(q) = p$
  - The one on the left is a local minimum
  - The one of the right is the global maximum
  - So the supply curve is given by the marginal cost curve above p^*