Risk and Uncertainty 2

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GR6211 - Microeconomic Analysis 1

Risk Aversion

- We motivated EU theory by appealing to risk aversion
- Does EU imply risk aversion?
- No!
- Consider someone who has $u(x) = x$
  - They will be risk neutral
- Consider someone who has $u(x) = x^2$
  - They will be risk loving
- So risk attitude has something to do with the shape of the utility function

Risk Aversion

- For this section we will think about lotteries with monetary prizes
- Let δ_x be the lottery that gives prize x for sure and $E(p)$ be the expected value of a lottery p

Definition

We say that a decision maker is risk averse if, for every lottery p

$$\delta_{E(p)} \succeq p$$

We say they are risk neutral if

$$\delta_{E(p)} \sim p$$

We say they are risk loving if

$$\delta_{E(p)} \preceq p$$

- We can say the same thing a different way

Definition

The **certainty equivalence** of a lottery p is the amount c such that

$$\delta_c \sim p$$

The risk premium is

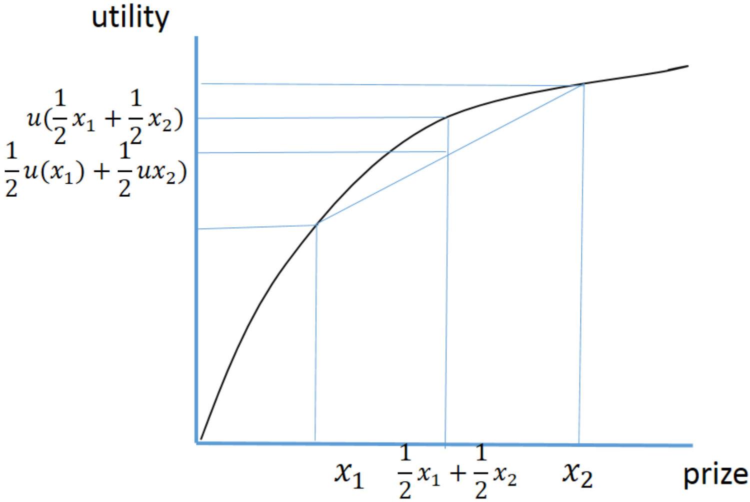
$$E(p) - c$$

Lemma

For a decision maker whose preferences are strictly monotonic in money

1. *They are risk averse if and only if for any p the risk premium is weakly positive*
2. *They are risk neurtal if and only if for any p the risk premium is zero*
3. *They are risk loving if and only if for any p the risk premium is weakly negative*

- We have made the claim that there is a link between risk aversion and the curvature of the utility function



- We can make this statement tight

Theorem

An expected utility maximizer

1. *Is risk averse if and only if u is concave*
2. *Is risk neutral if and only if u is linear*
3. *Is risk loving if and only if u is convex*

Proof.

Comes straight from Jensen's inequality: for a random variable x and a concave function u

$$E(u(x)) \leq u(E(x))$$

□

Measuring Risk Aversion

- We might want a way of measuring risk aversion from the utility function
- Intuitively, the more 'curvy' the utility function, the more risk averse
- How do we measure curvature?
- The second derivative $u''(x)$!
- Is this a good measure?
- No, because we can change the utility function in such a way that we don't change the underlying preferences, and change $u''(x)$

- One way round this problem is to use the **Arrow-Pratt** measure of **absolute** risk aversion

$$A(x) = \frac{-u''(x)}{u'(x)}$$

- This measure has some nice properties
  1. If two utility functions represent the same preferences then they have the same A for every x
  2. It measures risk aversion in the sense that the following two statements are equivalent
     - The utility function u has a higher Arrow Pratt measure than utility function v for every x
     - Utility function u gives a higher risk premium than utility function v for every p

- Why is it called a measure of **absolute** risk aversion?
- To see this, let's think of a function for which $A(x)$ is constant

$$u(x) = 1 - e^{-ax}$$

  - Note $u'(x) = ae^{-ax}$ and $u''(x) = -a^2 e^{-ax}$ so $A(x) = a$
- This is a constant absolute risk aversion (CARA) utility function

- Claim: for CARA utility functions, adding a constant amount to each lottery doesn't change risk attitues
- i.e if $\delta_x \succeq p$ then δ_{x+z} is preferred to a lottery p' which adds an amount z to each prize in p
- To see this note that

$$\begin{aligned}
u(x) &\geq \sum_y p(y)u(y) \\
1-e^{-ax} &\geq \sum_y p(y)\left(1-e^{-ay}\right) \\
&\Rightarrow 1-e^{-ax} \geq 1-\sum_y p(y)e^{-ay} \\
e^{-az}-e^{-ax}e^{-az} &\geq e^{-az}-\sum_y p(y)e^{-ay}e^{-az} \\
&\Rightarrow 1-e^{-a(x+z)} \geq \sum_y p(y)\left(1-e^{-a(y+z)}\right) \\
&\Rightarrow u(x+z) \geq \sum_y p(y)u(y+z)
\end{aligned}$$

Relative Risk Aversion

- Is this a sensible property?
- Maybe not
- Means that you should have the same attitude to a gamble between winning \$100 or losing \$75 whether you are a student earning \$20,000 a year or a professor earning millions!
- Perhaps a more useful measure is **relative** risk aversion

$$R(x) = xA(x) = -\frac{xu''(x)}{u'(x)}$$

Relative Risk Aversion

- An example of a Constant Relative Risk Aversion measure is

$$u(x) = \frac{x^{1-\rho} - 1}{1 - \rho}$$

- Note that $u'(x) = x^{-\rho}$, $u''(x) = -\rho x^{-\rho-1}$ and so $R(x) = \rho$
- CRRA utility functions have the property that proportional changes in prizes don't affect risk attitudes
- i.e if $\delta_x \succeq p$ then $\delta_{\alpha x}$ is preferred to a lottery p' which multiplies each prize in p by $\alpha > 0$

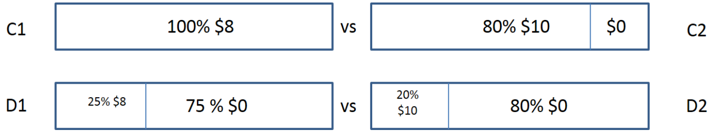
- To see this note that

$$
\begin{aligned}
u(x) &\geq \sum_y p(y)u(y) \\
&\Rightarrow \frac{x^{1-\rho}-1}{1-\rho} \geq \frac{\sum_y p(y)y^{1-\rho}-1}{1-\rho} \\
&\Rightarrow x^{1-\rho} \geq \sum_y p(y)y^{1-\rho} \\
&\Rightarrow \alpha^{1-\rho}x^{1-\rho} \geq \sum_y p(y)\alpha^{1-\rho}y^{1-\rho} \\
&\Rightarrow \frac{(\alpha x)^{1-\rho}-1}{1-\rho} \geq \frac{\sum_y p(y)\left(\alpha y\right)^{1-\rho}-1}{1-\rho} \\
u(\alpha x) &\geq \sum_y p'(y)u(y)
\end{aligned}
$$

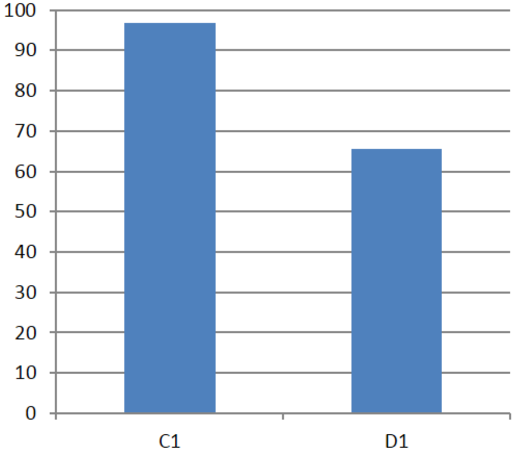
Are People Expected Utility Maximizers?

- Because of the work we have done above, we know what the ‘behavioral signature’ is of EU
  - The independence axiom
- Essentially this is picking up on the fact that EU demands preferences to be linear in probabilities
- Does this hold in experimental data?

The Common Ratio Effect



- What would you choose?
- Many people choose C1 and D2


100
90
80
70
60
50
40
30
20
10
0
C1
D1

The Common Ratio Effect

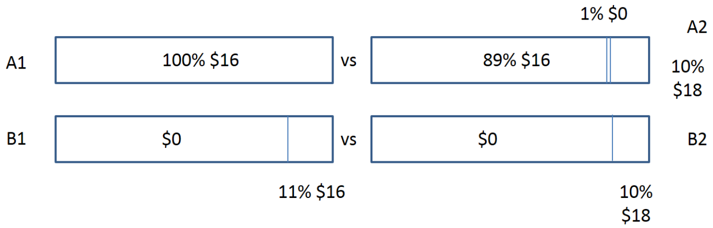
- This is a violation of the independence axiom
- Why?
- Because

$$\begin{aligned} D1 &= 0.25C1 + 0.75R \\ D2 &= 0.25C2 + 0.75R \end{aligned}$$

  where R is the lottery which pays 0 for sure
- Thus independence means that

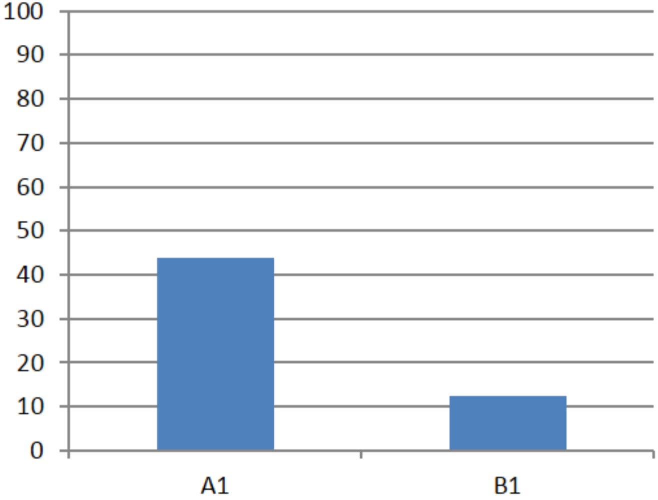
$$C1 \succeq C2 \Rightarrow D1 \succeq D2$$

The Common Consequence Effect



- What would you choose?
- Many people choose A1 and B2

The Common Consequence Effect



Explanations

- What do you think is going on?
- Many alternative models have been proposed in the literature
  - Disappointment: Gul, Faruk, 1991. "A Theory of Disappointment Aversion,"
  - Salience: Pedro Bordalo & Nicola Gennaioli & Andrei Shleifer, 2012. "Salience Theory of Choice Under Risk,"
- One of the most widespread and straightforward is **probability weighting**

Probability Weighting

- Maybe the problem that the Allais paradox highlights is that people do not 'believe' the probabilities that are told to them
  - For example they treat a 1% probability of winning $0 as if it is more likely than that
    - ‘I am unlucky, so the bad outcome is more likely to happen to me’
  - The difference between 0% and 1% seems bigger than the difference between 89% and 90%
- This is the idea behind the probability weighting model.

Simple Probability Weighting Model

- Approach 1: Simple probability weighting
- Let's start with expected utility

$$U(p) = \sum_{x \in X} p(x)u(x)$$

- And allow for probability weighting

$$V(p) = \sum_{x \in X} \pi(p(x))u(x)$$

  Where π is the probability weighting function
- This can explain the Allais paradox
  - For example if $\pi(0.01) = 0.05$

Simple Probability Weighting Model

- However, the simple probability weighting model is not popular
- For two reasons
  1. It leads to violations of stochastic dominance
  2. It doesn't really capture the idea of 'pessimism'

Pessimism

- Think back to the Allais paradox

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \succ \begin{pmatrix} 0.01 \\ 0.89 \\ 0.1 \end{pmatrix}$$

- It seems as if the 1% probability of $0 is being overweighted
- Is this just because it is a 1% probability?
- Or is it because it is a 1% probability **of the worst prize**
- If it is the latter, this is something that the simple probability weighting model cannot capture
  - Weights are only based on probability

- Consider the following two examples

Example

Lottery p :49% chance of \$10, 49% of winning \$0, 2% chance of winning \$5

Example

Lottery p :49% chance of \$10, 49% of winning \$0, 2% chance of losing \$1000

- Would you ‘weigh’ the 2% probability the same in each case?
  - Arguably not
  - If you were pessimistic then you might think that 2% is ‘more likely’ in the latter case than in the former
  - Can’t be captured by the simple probability weighting model

- Because of these two concerns, the simple probability weighting model is rarely used
- Instead people tend to use **rank dependent utility** (sometimes also called cumulative probability weighting)
- Probability weighting depends on
  - The **probability** of a prize
  - Its **rank** in the lottery - i.e. how many prizes are better or worse than it
- In practice this is done by applying weights **cumulatively**
- Here comes the definition
  - It looks scary, but don't panic!

Definition

A decision maker's preferences $\succeq$ over $\Delta(X)$ can be represented by a rank dependant utility model if there exists a utility function $u : X \rightarrow \mathbb{R}$ and a cumulative probability weighting function $\psi : [0,1] \rightarrow [0,1]$ such that $\psi(0) = 0$ and $\psi(1) = 1$, such that the function $U : \Delta(X) \rightarrow \mathbb{R}$ represents $\succeq$, where $U(p)$ is constructed in the following way:

1. The prizes of p are ranked $x_1, x_2, \ldots, x_n$ such that $x_1 \succ x_2 \cdots \succ x_n$
2. $U(p)$ is determined as

$$U(p) = \psi(p_1)u(x_1) + \sum_{i=2}^{n} \left(\psi\left(\sum_{j=1}^{i} p_j \right) - \psi\left(\sum_{k=1}^{i-1} p_k \right) \right) u(x_i)$$

- Let's go through an example: for prizes $10 > 5 > 0$ let p be equal to

$$\begin{pmatrix} 0.1 \\ 0.7 \\ 0.2 \end{pmatrix}$$

- How do we apply RDU?

- Well, first note that there are three prizes, so we can rewrite the expression above as

$$\begin{aligned} U(p) &= \psi(p_1)u(x_1) \\ &\quad + \left(\psi\left(p_1+p_2\right)-\psi\left(p_1\right)\right)u(x_2) \\ &\quad + \left(\psi\left(p_1+p_2+p_3\right)-\psi\left(p_1+p_2\right)\right)u(x_3) \end{aligned}$$

- The weight attached to the best prize is the weight of p_1
- The weight attached to the second best prize is the weight on the probability of
  - Getting something at least as good as the second prize
  - Minus the probability of getting something better than the second prize
  - And so on
- Notice that if ψ is the identity function this is just expected utility

- In this specific case

$$\begin{aligned} U(p) &= \psi(p_1)u(x_1) \\ &\quad + \left(\psi\left(p_1+p_2\right) - \psi\left(p_1\right)\right)u(x_2) \\ &\quad + \left(\psi\left(p_1+p_2+p_3\right) - \psi\left(p_1+p_2\right)\right)u(x_3) \end{aligned}$$

- Becomes

$$\begin{aligned} U(p) &= \psi(0.1)u(10) \\ &\quad + \left(\psi\left(0.8\right) - \psi\left(0.1\right)\right)u(5) \\ &\quad + \left(\psi\left(1\right) - \psi\left(0.8\right)\right)u(0) \end{aligned}$$

- In the first class we drew a distinction betweem
  - Circumstances of **Risk** (roulette wheels)
  - Circumstances of **Uncertainty** (horse races)
- So far we have been talking about roulette wheels
- Now horse races!

Risk vs Uncertainty

- Remember the key difference between the two
- Risk: Probabilities are **observable**
  - There are 38 slots on a roulette wheel
  - Someone who places a $10 bet on number 7 has a lottery with pays out $350 with probability 1/38 and zero otherwise
  - (Yes, this is not a fair bet)
- Uncertainty: Probabilities are **not observable**
  - Say there are 3 horses in a race
  - Someone who places a $10 bet on horse A does not necessarily have a 1/3 chance of winning
  - Maybe their horse only has three legs?

Subjective Expected Utility

- If we want to model situations of uncertainty, we cannot think about preferences over **lotteries**
- Because we don't know the probabilities
- We need a different set up
- We are going to thing about **acts**
- What is an act?

States of the World

- First we need to define **states of the world**
- We will do this with an example
- Consider a race between three horses
  - A(rchibald)
  - B(yron)
  - C(umberbach)
- What are the possible oucomes of this race?
  - Excluding ties

State	Ordering
1	A, B ,C
2	A, C, B
3	B, A, C
4	B, C, A
5	C, A, B
6	C, B, A

Acts

- This is what we mean by the states of the world
    - An exclusive and exhaustive list of all the possible outcomes in a scenario
- An **act** is then an action which is defined by the oucome it gives in each state of the world
- Here are two examples
    - Act f: A \$10 even money bet that Archibald will win
    - Act g: A \$10 bet at odds of 2 to 1 that Cumberbach will win

Acts

State	Ordering	Payoff Act f	Payoff Act g
1	A, B ,C	$10	-$10
2	A, C, B	$10	-$10
3	B, A, C	-$10	-$10
4	B, C, A	-$10	-$10
5	C, A, B	-$10	$20
6	C, B, A	-$10	$20

Subjective Expected Utility Theory

- So, how would you choose between acts f and g?
- SEU assumes the following:

1. Figure out the probability you would associate with each state of the world
2. Figure out the utility you would gain from each prize
3. Figure out the expected utility of each act according to those probabilities and utilities
4. Choose the act with the highest utility

- So, in the above example
- Utility from f :

$$\begin{aligned}&[\pi(ABC) + \pi(ACB)]\, u(10)\\&+ [\pi(BAC) + \pi(BCA)]\, u(-10)\\&+ [\pi(CBA) + \pi(CAB)]\, u(-10)\end{aligned}$$

  where π is the probability of each act
- Utility from g :

$$\begin{aligned}&[\pi(ABC) + \pi(ACB)]\, u(-10)\\&+ [\pi(BAC) + \pi(BCA)]\, u(-10)\\&+ [\pi(CBA) + \pi(CAB)]\, u(20)\end{aligned}$$

- Assuming utility is linear f is preferred to g if

$$\frac{[\pi(ABC) + \pi(ACB)]}{[\pi(CBA) + \pi(CAB)]} \geq \frac{3}{2}$$

- Or the probability of A winning is more than $3/2$ times the probability of C winning

Definition

Let X be a set of prizes, Ω be a (finite) set of states of the world and F be the resulting set of acts (i.e. F is the set of all functions $f : \Omega \rightarrow X$). We say that preferences $\succeq$ on the set of acts F has a subjective expected utility representation if there exists a utility function $u : X \rightarrow \mathbb{R}$ and probability function $\pi : \Omega \rightarrow [0,1]$ such that $\sum_{\omega\in\Omega} \pi(\omega) = 1$ and

$$\begin{aligned} f &\succeq g \\ &\Leftrightarrow \sum_{\omega\in\Omega} \pi(\omega) u\left(f(\omega)\right) \geq \sum_{\omega\in\Omega} \pi(\omega) u\left(g(\omega)\right) \end{aligned}$$

- Notes
  - Notice that we now have **two** things to recover: Utility and preferences
  - Axioms beyond the scope of this course: has been done twice - first by Savage[1] and later (using a trick to make the process a lot simpler) by Anscombe and Aumann[2]
  - Utility pinned down to positive affine transform
  - Probabilities are unique

[1] Savage, Leonard J. 1954. The Foundations of Statistics. New York, Wiley.

[2] Anscombe, F. J.; Aumann, R. J. A Definition of Subjective Probability. The Annals of Mathematical Statistics 34 (1963), no. 1, .

The Ellsberg Paradox

- Unfortunately, while simple and intuitive, SEU theory has some problems when it comes to describing behavior
- These problems are most elegantly demostrated by the Ellsberg paradox
  - This thought experiment has sparked a whole field of decision theory

The Ellsberg Paradox - A Reminder

- Choice 1: The 'risky bag'
  - Fill a bag with 20 red and 20 black tokens
  - Offer your subject the opportunity to place a $10 bet on the color of their choice
  - Then elicit the amount x such that the subject is indifferent between playing the gamble and receiving $\$x$ for sure.
- Choice 2: The 'ambiguous bag'
  - Repeat the above experiment, but provide the subject with no information about the number of red and black tokens
  - Then elicit the amount y such that the subject is indifferent between playing the gamble and receiving $\$y$ for sure.

- Typical finding
  - $x >> y$
  - People much prefer to bet on the risky bag
- This behavior cannot be explained by SEU?
- Why?

- What is the utility of betting on the risky bag?
- The probability of drawing a red ball is the same as the probability of drawing a black ball at 0.5
- So whichever act you choose to bet on, the utility of the gamble is

$$0.5u(\$10)$$

- What is the utility of betting on the ambiguous bag?
- Here we need to apply SEU
- What are the states of the world?
    - Red ball is drawn or black ball is drawn
- What are the acts?
    - Bet on red or bet on black

State	r	b
red	10	0
black	0	10

- How do we calculate the utility of these two acts?
  - Need to decide how likely each state is
  - Assign probabilities $\pi(r) = 1 - \pi(b)$
  - Note that these do **not** have to be 50%
  - Maybe you think I like red chips!

- Utility of betting on the red outcome is therefore

$$\pi(r)u(\$10)$$

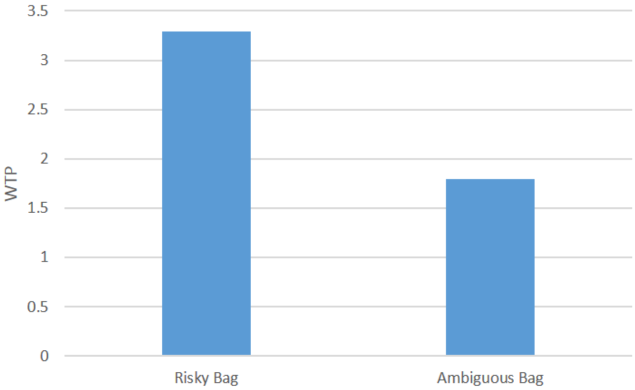
- Utility of betting on the black outcome is

$$\pi(b)u(\$10) = (1 - \pi(r))u(\$10)$$

- Because you get to choose which color to bet on, the gamble on the ambiguous urn is

$$\max\{\pi(r)u(\$10), (1 - \pi(r))u(\$10)\}$$

  - is equal to $0.5u(\$10)$ if $\pi(r) = 0.5$
  - otherwise is **greater** than $0.5u(\$10)$
  - should always (weakly) prefer to bet on the ambiguous urn
  - intuition: if you can choose what to bet on, 0.5 is the worst probability



- 61% of my last class exhibited the Ellsberg paradox
- For more details see *Halevy, Yoram. "Ellsberg revisited: An experimental study." Econometrica 75.2 (2007): 503-536.*

- So, as usual, we are left needing a new model to explain behavior
- There have been many such attempts since the Ellsberg paradox was first described
- We will focus on 'Maxmin Expected Utility' by Gilboa and Schmeidler[3]

[3] Gilboa, Itzhak & Schmeidler, David, 1989. "Maxmin expected utility with non-unique prior," Journal of Mathematical Economics, Elsevier, vol. 18(2), pages 141-153, April.

Maxmin Expected Utility

- Maxmin expected utility has a very natural interpretation....
- The world is out to get you!
  - Imagine that in the Ellsberg experiment was run by an evil and sneaky experimenter
  - After you have chosen whether to bet on red or black, they will increase your chances of losing
  - They will sneak some chips into the bag of the **opposite** color to the one you bet on
  - So if you bet on red they will put black chips in and visa versa

Maxmin Expected Utility

- How should we think about this?
- Rather than their being a single probability distribution, there is a **range** of possible distributions
- After you chose your act, you evaluate it using the **worst** of these distributions
- This is maxmin expected utility
  - you **maximize** the **minimum** utility that you can get across different probability distributions
- Has links to robust control theory in engineering
  - This is basically how you design aircraft

Definition

Let X be a set of prizes, Ω be a (finite) set of states of the world and F be the resulting set of acts (i.e. F is the set of all functions $f : \Omega \rightarrow X$). We say that preferences $\succeq$ on the set of acts F has a Maxmin expected utility representation if there exists a utility function $u : X \rightarrow \mathbb{R}$ and convex set of probability functions Π and

$$\begin{aligned} f &\succeq g \\ &\Leftrightarrow \min_{\pi \in \Pi} \sum_{\omega \in \Omega} \pi(\omega) u\left(f(\omega)\right) \geq \min_{\pi \in \Pi} \sum_{\omega \in \Omega} \pi(\omega) u\left(g(\omega)\right) \end{aligned}$$

Maxmin Expected Utility

- Maxmin expected utility can explain the Ellsberg paradox
  - Assume that $u(x) = x$
  - Assume that you think $\pi(r)$ is between 0.25 and 0.75
  - Utility of betting on the risky bag is $0.5u(x) = 5$
  - What is the utility of betting on red from the ambiguous bag?

$$\min_{\pi(r)\in[0.25,0.75]} \pi(r)u(\$10) = 0.25u(\$10) = 2.5$$

  - Similary, the utility from betting on black is

$$\min_{\pi(r)\in[0.25,0.75]} (1-\pi(r))\, u(\$10) = 0.25u(\$10) = 2.5$$

  - Maximal utility from betting on the ambiguous bag is lower than that from the risky bag