A Representation Theorem for Utility Maximization

Mark Dean

GR6211 - Microeconomic Analysis 1

A Representation Theorem

- When dealing with models that have latent (or unobservable) variables (such as utility maximization) we will want to find a *representation theorem*
- This consists of three things
  - A data set
  - A model
  - A set of conditions on the data which are **necessary** and **sufficient** for it to be consistent with the model
- A representation theorem tells us the observable implications of a model with unobservables
  - Means testing these conditions is the same as testing the model itself
- Often a representation theorem will have an associated **uniqueness result**
  - Tell us how precisely we have pinned down the unobservable variables

A Representation Theorem for Utility Maximization

- We are now going to develop a representation theorem for the model of utility maximization
- This is largely just formalizing the intuition we developed on the previous slides
- It is going to lead us to introduce a new model - that of preference maximization.

- The data we are going to use are the choices people make
- Notation:
  - X: **Finite** set of objects you might get to choose from
  - 2^X: The power set of X (i.e. all the subsets of X)
  - $\emptyset$: The empty set
- Our data is going to take the form of a **choice correspondence** which tells us what the person chose from each subset of X

Definition

A choice correspondence C is a mapping $C : 2^X/\emptyset \rightarrow 2^X/\emptyset$ such that $C(A) \subset A$ for all $A \in 2^X/\emptyset$.

Notes

- This is just a way of recording what we described previously
- For example, if we offered someone the choice of Jaffa Cakes and Kit Kats, and they chose Jaffa Cakes, we would write

$$C(\{kitkat, jaffacakes\}) = \{jaffacakes\}$$

- C is just a record of the choices made from all possible choice sets
  - i.e. all sets in 2^X apart from the empty set $\emptyset$
- We insist that the DM chooses something that was actually in the data set
  - i.e. $C(A) \subset A$
- **Important**: Choice correspondence is non-empty: something is chosen from each choice set

Notes

- What are some issues with this data set?

1. X Finite
2. Observe choices from all choice sets
3. We allow for people to choose more than one option!
    - i.e. we allow for data of the form

$$C(\{kitkat, jaffacakes, lays\}) = \{jaffacakes, kitkat\}$$

    - Which we interpret as something like "the decision maker would be happy with either jaffa cakes or lays from this choice set"

- These assumptions make our life easier, but are undesirable
    - We will relax them in later lectures

Notes

- Also, note that we are implicitly assuming that choice *only depends on the elements in A*
- Not (for example)
    - The order in which they are presented
    - A reference point
    - The amount of time people have to think
    - etc.
- We will come back to this when we discuss some of the evidence for and against utility maximization

Utility Maximization

- The model we want to test is that of utility maximization
- i.e. there exists a utility function $u : X \rightarrow \mathbb{R}$
- Such that the things that are chosen are those which maximize utility
  - For every A
$$C(A) = \arg\max_{x \in A} u(x)$$
- If this is true, we say that u **rationalizes** C
- If C can be rationalized by some u then we say it has a **utility representation**

- We want to know when data is consistent with utility maximization
  - i.e. it has a utility representation
- So we would like to find a set of conditions on C such that it has a utility representation **if and only if** these conditions are satisfied
  - Testing these conditions is then the same as testing the model of utility maximization

- You may remember a condition called the Weak Axiom of Revealed Preference from Intermediate Micro

$$\text{If } x, y \in A \cap B,\ x \in C(A) \text{ and } y \in C(B) \Rightarrow x \in C(B)$$

- We will break WARP down into two parts

Axiom α (AKA Independence of Irrelevant Alternatives) If $x \in B \subseteq A$ and $x \in C(A)$, then $x \in C(B)$

Axiom β If $x, y \in C(A)$, $A \subseteq B$ and $y \in C(B)$ then $x \in C(B)$

- You can (and will) show that α and β are equivalent to WARP
  - i.e. a data set satisfies α and β iff it satisfies WARP
  - α is 'from large to small'
  - β is 'from small to large'
- Notice we can **test** these conditions!
- If we have data, we can see if they are satisfied

- These conditions form the basis of our first representation theorem

Theorem

A Choice Correspondence on a finite X has a utility representation **if and only if** *it satisfies axioms α and β*

- **if**: if α and β are satisfied then a utility representation exists
- **only if:** if a utility representation exists then α and β are satisfied

- We are going to prove this theorem
- Before we do so, we are going to introduce the notion of **preferences**, and the associated model of **preference maximization**
- Will explain why after we have introduced the model

The Questionnaire

- Consider the alternatives in X
  - e.g. Jaffa cakes, Kit kat, Lays
- Consider an exhaustive list of questions:

  Do you like alternative x as much as alternative y?

- If the answer is yes, then we write $x \succeq y$

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Do you like...	Answer	We write...
j as much as j	yes	$j \succeq j$
k as much as k	yes	$k \succeq k$
l as much as l	yes	$l \succeq l$
j as much as k	yes	$j \succeq k$
k as much as j	no	
j as much as l	yes	$j \succeq l$
l as much as j	no	
k as much as l	yes	$k \succeq l$
l as much as k	no	

The Questionnaire

- Where do these preferences come from?
  - Could be choices (we will come back to this)
  - But we could ask people to express preferences over objects that we couldn't actually give them....
- Note that this is slightly different from the definition of questionnaire Q in Rubinstein's book
  - In fact it is his questionnaire R

The Questionnaire

- Technically speaking $\succeq$ is a **binary relation**

Definition

Consider a set X and denote by $X \times X$ its Cartesian Product. A binary relation B on X is a subset of $X \times X$. We write $B \subseteq X \times X$ and xBy if $(x, y) \in B$.

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    - Example: for

	l	k	j
l	✓	✓	✓
k		✓	✓
l			✓

- is equivalent to

$$jBj, jBl, jBk, kBk, kBl, lBl$$

- Examples of other binary relations
    - $X = \mathbb{R}$, $B = \geq$
    - $X =$population of New York, B="works with"

The Questionnaire

- Should we allow any possible answers to the questionnaire?
- No! Or at least we are going to rule some things out.
  - You cannot answer 'I don't know' or 'I like x much more than y' (only yes or no answers)
  - You have to answer 'yes' at least one of the questions
    - Do you like alternative x as much as alternative y?
  - or
    - Do you like alternative y as much as alternative x?
  - Coherence
    - If you like x as much as y and y as much as z you must say that you like x as much as z

The Questionnaire

- Do these seem like sensible properties?
  - First, what do we mean by 'sensible'?
  - Normative vs Positive statements
- Possible issues
  - Do you prefer coffee with 1 grain of sugar to 0 grains of sugar in your coffee?
  - Do you prefer a sun hat to a rain coat?
  - Do you prefer txuleta or oilasko for dinner?
  - Aggregation:

	A	*B*	*C*
1st	j	k	l
2nd	k	l	j
3rd	l	j	k

  - Majority rule will lead to a violation of transitivity (a **Condorcet cycle)**

- These restrictions mean that the binary relation $\succeq$ has certain properties
  - Completeness: for every x and y in X either $x \succeq y$ or $y \succeq x$ (or both)
  - Transitivity: if $x \succeq y$ and $y \succeq z$ then $x \succeq z$
  - Reflexivity: $x \succeq x$ for all $x \in X$
- There are many other properties one can define on binary relations, for example
  - Antisymmetric: $xRyRx$ implies $x = y$
  - Asymmetric: If xRy then not yRx
  - Symmetry: xRy implies yRx
- Under what circumstances would $\succeq$ have these properties?

- Let X be a non-empty set and R a binary relation on X

Definition
If R is transitive and reflexive then it is a **preorder**. If it is also antisymmetric it is a **partial order.** If it is also complete it is a **linear order**

Definition
(X, R) is a **preordered set** if R is a preorder, a **poset** if R is a partial order and a **loset** if R is a linear order

Definition
We will say R is a **preference relation** if it is a complete preorder

- Note that some people (mainly weird decision theorists) will use preference relation to refer to a preorder

Preference Relations

- Notice that we can use $\succeq$ to define other binary relations:
  - Strict Preference

$$x \succ y: \text{ if } x \succeq y \text{ but not } y \succeq x$$

  - This is called the **asymmetric** part of $\succeq$
  - Indifference

$$x \sim y: \text{if } x \succeq y \text{ and } y \succeq x$$

  - This is called the **symmetric** part of $\succeq$
- What properties do these binary relations have?
  - Complete?
  - Transitive?
  - Asymmetric?
  - Symmetric?

- We can use preferences to form a model of choice
- We say that the complete preference relation $\succeq$ represents a choice function C if, for every A

$$C(A) = \{x \in A | x \succeq y \text{ for all } y \in A\}$$

  - i.e. the things that are chosen are those that are preferred to everything else in the choice set
- Note $\{x \in A | x \succeq y \ \forall y \in A\}$ are the $\succeq$-maximal elements in A
  - If X is finite can we guarantee the existence of $\succeq -$maximal elements?

But Why?

- I hope you agree that the above concepts are well defined
- But why do we want to introduce the idea of preferences and preference maximization?

1. Preference maximization is in some sense a more 'honest' model
    - Will come back to this, but basically preferences provide a unique representation of choice, while utility does not
2. It is often convenient to treat preferences as data
    - Preferences may in fact be the primitive
    - Even if not, translation from choice to preference relatively straightforward
    - When dealing with more complex models of choice, it can be easier to start with the assumption of a well behaved preference relation, the add further conditions
    - Will see this when we talk about expected utility theory
3. Introducing preferences will help us prove our representation theorem for utility maximization

- We are now going to use the concept of preferences to prove our representation theorem for utility
- In doing so we are going to link together choice, preferences, and utility
- We have already seen how we will link choice and preferences
- To link preferences and utility we can treat preferences as data and prove representation theorems of that type
- We say that a utility function u **represents** preferences $\succeq$ if

$$\begin{aligned} u(x) &\geq u(y) \text{ if and only if} \\ x &\succeq y \end{aligned}$$

- In fact, this is how we are going to prove our representation theorem
- If we can find
  - A preference relation which represents choices
  - A utility function which represents preferences

  we are done!
- Preferences represents choices means

$$C(A) = \{x \in A | x \succeq y \text{ for all } y \in A\}$$

- Utility represents preferences means

$$u(x) \geq u(y) \Longleftrightarrow x \succeq y$$

- So

$$\begin{aligned} C(A) &= \{x \in A | u(x) \geq u(y) \text{ for all } y \in A\} \\ &= \arg\max_{x \in A} u(x) \end{aligned}$$

- Thus, in order to prove that axioms α and β are equivalent to utility maximization we will do the following

1. Show that if the data satisfies α and β then we can find a preference relation $\succeq$ which represents the data
2. Show that if a binary relation is complete and transitive then we can find a utility function u which represents them
3. Show that if the data has a utility representation then it must satisfy α and β (this you will do for homework)

Theorem

Let C be a choice correspondence on a set X. Then there exists a preference relation $\succeq$ which represents C - i.e.

$$C(A) = \{x \in A | x \succeq y \text{ for all } y \in A\}$$

if and only if C satisfies axioms α and β

Proof.

Sufficiency: (Sketch - details in class):

1. Define candidate relation $\trianglerighteq$ using binary choice
2. Show that $\trianglerighteq$ is a preference relation
3. Show that $\trianglerighteq$ represents choice in all choice sets

□

Proof.

Necessity - Postponed for later □

Theorem

Let $\succeq$ be a binary relation on a **finite** *set X. Then there exists a utility function $u : X \to \mathbb{R}$ which represents $\succeq$: i.e.*

$$\begin{aligned} u(x) &\geq u(y) \text{ if and only if} \\ x &\succeq y \end{aligned}$$

if and only if $\succeq$ is a preference relation

Proof.

Sufficiency: (Sketch - details in class):

1. Proof by induction on the size of the set X
2. Obviously true of $|X| = 1$
3. For $|X| = N$, remove one item x, and by induction let v be a utility representation on $X/\{x\}$
4. Show that we can find a number to assign to x which completes a utility representation for X

- For homework you will show that if a choice correspondence has a utility representation then it must satisfy α and β
- Note that, with the proofs we have just done, this means that we have proved our main theorem

Theorem

A Choice Correspondence on a finite X has a utility representation **if and only if** *it satisfies axioms α and β*

Comments

- Now we have proved this theorem let me provide some commentary

1. **Properly specifying alternatives:**
    - The following looks like a violation of α, but is it 'irrational'?

$$\begin{aligned} C(\textit{steak tatre, chicken, frogs legs}) &= \textit{steak tatre} \\ C(\textit{steak tatre, chicken,}) &= \textit{chicken} \end{aligned}$$

2. **Do not over interpret**
    - If someone's choices satisfy WARP, does this mean that they are maximizing utility?
3. **What are the advantages of providing the representation theorem?**
    - Testability
    - Providing an understanding of the model
    - Allow us to compare different models more easily
    - Question: Are all axioms testable?

How Unique is Our Utility Function?

- We now know that if α and β are satisfied, we can find **some** utility function that will explain choices
- Is it the only one?

How Unique is Our Utility Function?

Croft's Choices	
Available Snacks	**Chosen Snack**
Jaffa Cakes, Kit Kat	Jaffa Cakes
Kit Kat, Lays	Kit Kat
Lays, Jaffa Cakes	Jaffa Cakes
Kit Kat, Jaffa Cakes, Lays	Jaffa Cakes

- These choices could be explained by $u(J) = 3$, $u(K) = 2$, $u(L) = 1$
- What about $u(J) = 100000$, $u(K) = -1$, $u(L) = -2$?
- Or $u(J) = 1$, $u(K) = 0.9999$, $u(L) = 0.998$?

How Unique is Our Utility Function?

- In fact, if a data set obeys α and β there will be **many** utility functions which will rationalize the data

Theorem

Let $u : X \to \mathbb{R}$ be a utility representation for a Choice Correspondence C. Then $v : X \to \mathbb{R}$ will also represent C if and only if there is a strictly increasing function T such that

$$v(x) = T(u(x)) \; \forall \; x \in X$$

How Unique is Our Utility Function?

Snack	u	v	w
Jaffa Cake	3	100	4
Kit Kat	2	10	2
Lays	1	-100	3

- v is a strictly increasing transform on u, and so represents the same choices
- w is not, and so doesn't
  - For example think of the choice set $\{k, l\}$
  - u says they should choose kit cat
  - w says they should choose lays

Why Does This Matter?

- It is important that we know how much the data can tell us about utility
  - This is equivalent to figuring out identification in econometrics
  - How well does our data identify utility?
- For example, our results tell us that there **is** a point in designing a test to tell whether people maximize utility
- But there is **no** point in designing a test to see whether the utility of Kit Kats is **twice** that of Lays
  - Assuming α and β is satisfied, we can always find a utility function for which this is true
  - And another one for which this is false!
- We can use choices to help us determine that the utility of Kit Kats is higher than the utility of Lays
- But nothing in our data tells us **how much** higher is the utility of Kit Kats

Why Does This Matter?

- Question: what is the equivalent uniqueness statement for the model of preference maximization?