Utility Maximization 2: Extensions

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GR6211 - Microeconomic Analysis 1

- We have now proved the following theorem

Theorem

A Choice Correspondence on a finite X has a utility representation **if and only if** *it satisfies axioms α and β*

- Great! We know how to test the model of utility maximization!
- However, our theorem is only as useful as the data set we are working with
- As discussed at the time, there are some problems with the data we have assumed so far

Problems with the Data Set

- What are some issues with this data set?

1. Observe choices from all choice sets
2. We allow for people to choose more than one option
   - i.e. we allow for data of the form

$$C(\{kitkat, jaffacakes, lays\}) = \{jaffacakes, kitkat\}$$

3. X Finite

What if X is not Finite?

- So far we have assumed that the set of available alternatives is finite

Theorem

A Choice Correspondence on a **finite** *X has a utility representation if and only if it satisfies axioms α and β*

- However, this may be limiting
  - Choice from lotteries
  - Choice from budget sets
- Can we drop the word 'finite' from the above theorem?

What if X is not Finite?

- Remember we proved the theorem in three steps
  1. Show that if the data satisfies α and β then we can find a complete, transitive, reflexive preference relation $\succeq$ which represents the data
  2. Show that if the preferences are complete, transitive and reflexive then we can find a utility function u which represents them
  3. Show that if the data has a utility representation then it must satisfy α and β
- Where did we make use of finiteness?

What if X is not Finite?

- In fact the problems relating choice to preference maximization are relatively minor
- The main issue here is that, if we want to define choice on **all** subsets of X we cannot guarantee that

$$C(A) = \{x \in A | x \succeq y \text{ for all } y \in A\}$$

  is well defined
  - Example?
- But we can get round this relatively easily
  - For example by demanding that we only observe choices from **finite** subsets of X
  - Even if X itself is not finite
  - As we shall see later we may be able to do better than this

What if X is not Finite?

- What about the relationship between preference and utility?
- Here in the proof we made heavy use of finiteness
  - Induction
- Are we in trouble?
- Just because we made use of the fact that X was finite in that particular proof doesn't mean that it is necessary for the statement to be true
- Maybe we will be lucky and the statement remains true for arbitrary X....
- Sadly not

- Some definitions you should know

Definition

The natural, or counting numbers, denoted by $\mathbb{N}$, are the set of numbers $\{1, 2, 3, \ldots\ldots\}$

Definition

The integers, denoted by $\mathbb{Z}$, are the set of numbers $\{\ldots, -3, -2, -1, 0, 1, 2, 3, ..\}$

Definition

The rational numbers, denoted by $\mathbb{Q}$, are the set of numbers

$$\mathbb{Q} = \left\{ \frac{a}{b} | a \in \mathbb{Z},\ b \in \mathbb{N} \right\}$$

Definition

A set is *countably infinite* if there is a bijection between that set and the natural numbers

- Here are some properties of $\mathbb{Q}$ and $\mathbb{R}$.

1. $\mathbb{Q}$ is countable
2. $\mathbb{R}$ is uncountable
3. For every $a, b \in \mathbb{R}$ such that $a < b$, there exists a $c \in \mathbb{Q}$ such $a < c < b$ (i.e. $\mathbb{Q}$ is dense in $\mathbb{R}$)

Definition

Let $\succeq$ be a binary relation on $\mathbb{R} \times \{1, 2\}$ such that

$$\begin{aligned} \{a, b\} &\succeq \{c, d\} \text{ iff} \\ \text{(i) } a &> c \\ \text{or (ii) } a &= c \text{ and } b \geq d \end{aligned}$$

You should check that you agree that $\succeq$ is a complete preference relation.

Fact

There is no utility function that rationalizes $\succeq$.

Utility Representation with Non-Finite X

- So what can we do in order to ensure that preferences have a utility representation?
- First things first: how big is the problem?
- The counter example above made use of the fact that X was uncountable
- Does this mean the problem goes away if X is **countably** finite?
- It turns out the answer is yes

Theorem

If a relation $\succeq$ on a **countable** *X is complete, transitive and reflexive then there exists a utility function $u : X \rightarrow \mathbb{R}$ which represents $\succeq$, i.e.*

$$u(x) \geq u(y) \Longleftrightarrow x \succeq y$$

Utility Representation with Uncountable X

- We know from the example of lexicographic preferences that we cannot replace ‘countable’ with ‘any’ X in the previous theorem
- In order to guarantee that we have a utility representation of a preference relation on an uncountable X we need another condition

- One way to go is to insist that preferences are **continuous**
- Broadly speaking, this means that if we change the items a little bit the preferences also change only a little bit
- i.e. they don't 'jump'

Definition

We say that a preference relation $\succeq$ on a **metric space** X is continuous if, for any $x, y \in X$ such that $x \succ y$, there exists an $\varepsilon > 0$ such that, for any $x' \in B(x, \varepsilon)$ and $y' \in B(y, \varepsilon)$, $x' \succ y'$

- Examples of preferences that are not continuous?
    - Lexicographic preferences
    - 'The price is right'

- An alternative characterization of continuity:

Lemma

A preference relation $\succeq$ on a metric space X is continuous if and only if, for every $x, y \in X$ and sequence $\{x_n, y_n\}$ such that $x_n \rightarrow x$ and $y_n \rightarrow y$ then $x_n \succeq y_n \; \forall \; n$ implies $x \succeq y$

- *i.e. the graph of $\succeq$ is closed*

- You will prove for homework that these two definitions are equivalent

- One thing that is relatively easy to prove is that continuity of utility implies continuity of preference

Theorem

If a preference relation $\succeq$ can be represented by a continuous utility function then it is continuous

- One of the most famous theorems in mathematical social sciences is that continuity guarantees the existence of a continuous utility representation

Theorem (Debreu)

Let X be a separable metric space, and $\succeq$ be a complete preference relation on X. If $\succeq$ is continuous, then it can be represented by a continuous utility function.

- Proving this in all its glory is beyond us, so we are going to prove something weaker

Theorem

Let X be a convex subset of $\mathbb{R}^n$ and $\succeq$ be a complete preference relation on X. If $\succeq$ is continuous, then it can be represented by a utility function.

- So now we have a method of dealing with utility and preferences in uncountable domains
- What about choice?
- Here we now have two issues
    1. We need to guarantee that maximal elements exist in all choice sets
    2. We would like to make sure the preferences that represent choices are continuous

Back to Choice

- To deal with problem 1 we will restrict ourselves to **compact** subsets of X
- Notice that if we can guarantee continuous preferences then this solves the first problem
  - Continuous preferences are equivalent to continuous utility functions
  - Continuous functions on compact sets obtain their maximum
- So how can we guarantee choice can be represented by continuous preferences?
- We would like choices to be continuous!
  - Choice sets that are ‘close’ to each other give rise to ‘similar’ choices

- How can we make this formal?
- We need a metric on sets!

Definition (The Hausdorff metric)

Let (X, d) be a metric space, and $cb(X)$ be the set of all closed and bounded subsets of X. We will define the metric space $(cb(X), d^h)$, where d^h is the Hausdorff metric induced by d, and is defined as follows: For any $A, B \in cb(X)$, define $\Lambda(A, B)$ as $\sup_{x \in A} d(x, B)$. Now define

$$d^H(A, B) = \max\{\Lambda(A, B), \Lambda(B, A)\}$$

- We can use this to define a **continuous choice correspondence**

Definition

Let X be a compact metric space and Ω_X be the set of all closed subsets of X and $C : \Omega_X \to 2^X$ be a choice correspondence. If $S_m \to S$ for $S_m, S \in \Omega_X$, $x_m \in C(S_m)$ $\forall$ m and $x_m \to x$, implies that $x \in C(S)$, then we say C is continuous.

- It turns out that continuity, plus α and β, is enough to give us our desired results

Theorem

Let X be a compact metric space and Ω_X be the set of all closed subsets of X and $C : \Omega_X \to 2^X$ be a choice correspondence. C satisfies properties α , β and continuity if and only if there is a complete, continuous preference relation $\succeq$ on X that rationalizes C.

Choices from all Choice Sets?

- Imagine running an experiment to try and test α and β
- The data that we need is the choice correspondence

$$C : 2^X / \emptyset \rightarrow 2^X / \emptyset$$

- How many choices would we have to observe?
- Lets say $|X| = 10$
  - Need to observe choices from every $A \in 2^X / \emptyset$
  - How big is the power set of X?
  - If $|X| = 10$ need to observe 1024 choices
  - If $|X| = 20$ need to observe 1048576 choices
- This is not going to work!

Choices from all Choice Sets?

- So how about we forget about the requirement that we observe choices from all choice sets
- Are α and β still enough to guarantee a utility representation?

$$
\begin{aligned}
C(\{x,y\}) &= \{x\} \\
C(\{y,z\}) &= \{y\} \\
C(\{x,z\}) &= \{z\}
\end{aligned}
$$

- If this is our only data then there is no violation of α or β
- But no utility representation exists!
- We need a different approach!

- In order to do this we are going to have to know a few more things about order theory (the study of binary relations)
- In particular we are going to need some definitions

Definition

A transitive closure of a binary relation R is a binary relation $T(R)$ that is the smallest transitive binary relation that contains R.

- i.e. $T(R)$ is
  - Transitive
  - Contains R in the sense that xRy implies $xT(R)y$
  - Any binary relation that is smaller (in the subset sense) is either intransitive or does not contain R
- Example?
- Question: is this always well defined?

- We can alternatively define the transitive closure of a binary relation R on X as the following:

Remark

- 1. *Define* $R_0 = R$
  2. *Define* R_m *as* $xR_m y$ *if there exists* $z_1, ..., z_m \in X$ *such that* $xRz_1 R...Rz_m Ry$
  3. $T = R \cup_{i \in \mathbb{N}} R_m$

Definition

Let $\succeq$ be a preorder on X. An **extension** of $\succeq$ is a preorder $\trianglerighteq$ such that

$$\succeq \quad \subset \trianglerighteq$$
$$\succ \quad \subset \triangleright$$

Where

- $\succ$ is the asymmetric part of $\succeq$, so $x \succ y$ if $x \succeq y$ but not $y \succeq x$
- $\triangleright$ is the asymmetric part of $\trianglerighteq$, so $x \triangleright y$ if $x \trianglerighteq y$ but not $y \trianglerighteq x$

- Example?

- We are also going to need one theorem

Theorem (Sziplrajn)

For any nonempty set X and preorder $\succeq$ on X there exists a complete preorder that is an extension of $\succeq$

- Relatively easy to prove if X is finite, but also true for any arbitrary X

- Okay, back to choice
- The approach we are going to take is as follows:
  - Imagine that the model of preference maximization is correct
  - What observations in our data would lead us to conclude that x was preferred to y?

- We say that x is **directly revealed preferred to** y $(xR^D y)$ if, for some choice set A

$$\begin{aligned} y &\in A \\ x &\in C(A) \end{aligned}$$

- We say that x is **revealed preferred to** y (xRy) if we can find a set of alternatives w_1, $w_2, \ldots .w_n$ such that
  - x is directly revealed preferred to w_1
  - w_1 is directly revealed preferred to w_2
  - ...
  - w_{n-1} is directly revealed preferred to w_n
  - w_n is directly revealed preferred to y
- I.e. R is the transitive closure of R^D

- We say x is **strictly revealed preferred to** y (xSy) if, for some choice set A

$$\begin{aligned} y &\in A \textbf{ but not } y \in C(A) \\ x &\in C(A) \end{aligned}$$

Notes

- Is it always true that choosing x over y means that you prefer x to y?
- Almost certainly not
  - Think of a model of 'consideration sets'
- Only true in the context of the model of preference maximization

The Generalized Axiom of Revealed Preference

- Note that we can observe revealed preference and strict revealed preference from the data
- With these definitions we can write an axiom to replace α and β
- What behavior is ruled out by utility maximization?

Definition

A choice correspondence C satisfies the Generalized Axiom of Revealed Preference (GARP) if it is never the case that x is revealed preferred to y, and y is **strictly** revealed preferred to x

- i.e. xRy implies not ySx

The Generalized Axiom of Revealed Preference

Theorem

A choice correspondence C on an arbitrary subset of $2^X/\oslash$ satisfies GARP if and only if it has a preference representation

Corollary

A choice correspondence C on an arbitrary subset of $2^X/\oslash$ with X finite satisfies GARP if and only if it has a utility representation

Choices from all Choice Sets?

- Note that this data set violates GARP

$$\begin{aligned} C(\{x,y\}) &= \{x\} \\ C(\{y,z\}) &= \{y\} \\ C(\{x,z\}) &= \{z\} \end{aligned}$$

- $xR^D y$ and $yR^D z$ so xRz
- But zSx

Choice Correspondence?

- Another weird thing about our data is that we assumed we could observe a choice **correspondence**
  - Multiple alternatives can be chosen in each choice problem
- This is not an easy thing to do!
- What about if we only get to observe a choice function?
  - Only one option chosen in each choice problem
- How do we deal with indifference?

Choice Correspondence?

- One of the things we could do is assume that the decision maker chooses **one of** the best options

$$C(A) \in \arg\max_{x \in A} u(x)$$

- Is this going to work?
- No!
- Any data set can be represented by this model
  - Why?
  - We can just assume that all alternatives have the same utility!

Choice Correspondence?

- Another thing we can do is assume away indifference

$$C(A) = \arg\max_{x \in A} u(x)$$

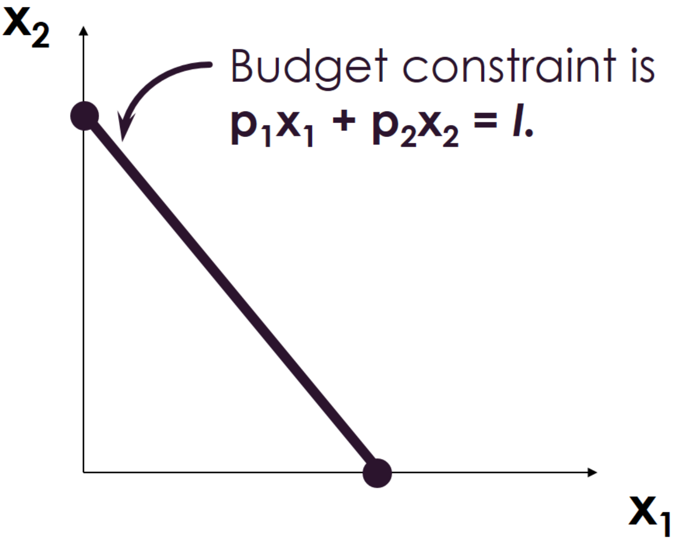
  - for some one-to-one function u
- Is this going to work?
- Yes
  - Implies that data is a function
  - Property α (or GARP) will be necessary and sufficient (if X is finite)
- But maybe we don't **want** to rule out indifference!
  - Maybe people are sometimes indifferent!

- Need some way of identifying when an alternative x is **better than** alternative y
  - i.e. some way to identify strict preference
- One case in which we can do this is if our data comes from people choosing **consumption bundles** from **budget sets**
  - Should be familiar from previous economics courses
- The objects that the DM has to choose between are bundles of different commodities

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

- And they can choose any bundle which satisfies their budget constraint

$$\left\{ x \in \mathbb{R}^n_+ \middle| \sum_{i=1}^{n} p_i x_i \leq I \right\}$$



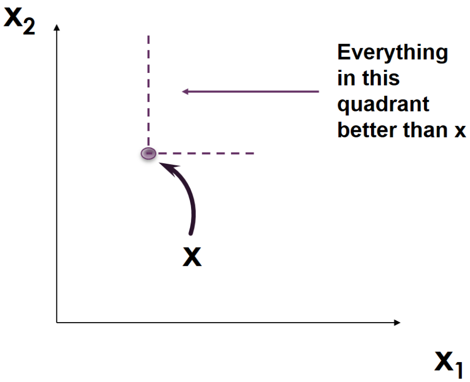
x_2
Budget constraint is
$p_1x_1 + p_2x_2 = I$.
x_1

Monotonicity

- Claim: We can use choice from budget sets to identify strict preference
  - Even if we only see a single bundle chosen from each budget set
- **As long as** we assume something about how preferences work
- One example: More is better

$$\begin{aligned} x_n &\geq y_n \text{ for all } n \text{ and } x_n > y_n \text{ for some } n \\ \text{implies that } x &\succ y \end{aligned}$$

  - i.e. preferences are **strictly monotonic**


x_2
Everything
in this
quadrant
better than x
x
x_1

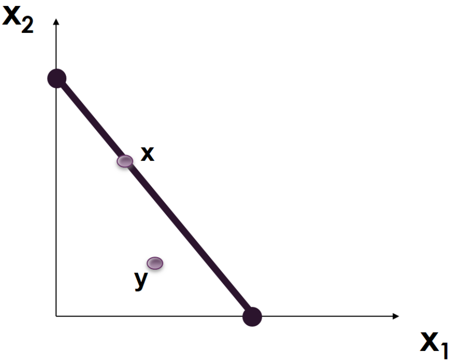
Monotonicity

- Claim: if p^x is the prices at which the bundle x was chosen

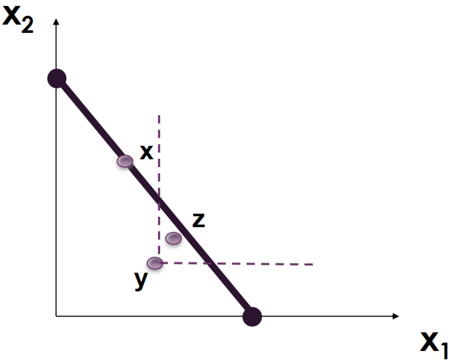
$$p^x x > p^x y \text{ implies } x \succ y$$

  - Why?

Revealed Strictly Preferred



- Because x was chosen, we know $x \succsim y$
- Because $p^x x > p^x y$ we know that y was **inside** the budget set when x was chosen
- Could it be that $y \succsim x$?



- Because y is inside the budget set, there is a z which is better than y **and** affordable when x was chosen
- Implies that $x \succsim z$ and (by monotonicity) $z \succ y$
- By transitivity $x \succ y$

- In fact we can make use of a weaker property than strict monotonicity

Definition

We say preferences $\succsim$ are **locally non-satiated** on a metric space X if, for every $x \in X$ and $\varepsilon > 0$, there exists

$$y \in B(x, \varepsilon)$$

such that

$$y \succ x$$

Lemma

Let x^j and x^k be two commodity bundles such that $p^j x^k < p^j x^j$. If the DM's choices can be rationalized by a complete locally non-satiated preference relation, then it must be the case that $x^j \succ x^k$

- When dealing with choice from budget sets we say
  - x is **directly revealed preferred to** y if $p^x x \geq p^x y$
  - x is **revealed preferred to** y if we can find a set of alternatives w_1, $w_2, \ldots . w_n$ such that
    - x is directly revealed preferred to w_1
    - w_1 is directly revealed preferred to w_2
    - ...
    - w_{n-1} is directly revealed preferred to w_n
    - w_n is directly revealed preferred to y
  - x is **strictly revealed preferred to** y if $p^x x > p^x y$

Theorem (Afriat)

Let $\{x^1, ...x^I\}$ *be a set of chosen commodity bundles at prices* $\left\{p^1, ..., p^I\right\}$. *The following statements are equivalent:*

1. *The data set can be rationalized by a locally non-satiated set of preferences* $\succeq$ *that can be represented by a utility function*
2. *The data set satisfies GARP (i.e.* xRy *implies not* ySx*)*
3. *There exists positive* $\left\{u^i, \lambda^i\right\}_{i=1}^{I}$ *such that*

$$u^i \leq u^j + \lambda^j p^j (x^i - x^j) \;\forall\; i, j$$

4. *There exists a continuous, concave, piecewise linear, strictly monotonic utility function* u *that rationalizes the data*

Things to note about Afriat's Theorem

- Compare statement 1 and statement 4
  - The data set can be rationalized by a locally non-satiated set of preferences $\succeq$ that can be represented by a utility function
  - There exists a continuous, concave, piecewise linear, strictly monotonic utility function u that rationalizes the data
- This tells us that there is no empirical content to the assumptions that utility is
  - Continuous
  - Concave
  - Piecewise linear
- If a data set can be rationalized by any locally non-satiated set of preferences it can be rationalized by a utility function which has these properties

Things to note about Afriat's Theorem

- What about statement 3?
  - There exists positive $\left\{u^i, \lambda^i\right\}_{i=1}^{I}$ such that

$$u^i \leq u^j + \lambda^j p^j (x^i - x^j) \ \forall \ i, j$$

- This says that the data is rationalizable if a certain linear programming problem has a solution
  - Easy to check computationally
  - Less insight than GARP
  - But there are some models which do not have an equivalent of GARP but do have an equivalent of these conditions

Things to note about Afriat's Theorem

- Where do these conditions come from?
- Imagine that we knew that this problem was differentiable

$$\max u(x) \text{ subject to } \sum_j p_j^i x_j \leq I$$

with u concave

- FOC for every problem i and good j

$$\frac{\partial u(x^i)}{\partial x_j^i} = \lambda^i p_j^i$$

- Implies

$$\nabla u(x^i) = \lambda^i p^i$$

  - where ∇u is the gradient function, p^i is the vector of prices and λ^i the lagrnge multiplier

Things to note about Afriat's Theorem

- Recall (or learn), that for concave functions

$$u(x^i) \leq u(x^j) + \nabla u(x^j)(x^i - x^i)$$

  - i.e. function lies below the tangent
- So

$$u(x^i) \leq u(x^j) + \lambda^j p^j (x^i - x^j)$$