Constrained Optimization Solutions[1]

Math Camp 2012

1 Exercises

1. There are two commodities: x and y. Let the consumer's consumption set be $\mathbb{R}^2_+$ and his preference relation on his consumption set be represented by $u(x,y) = -(x-4)^2 - y^2$. When his initial wealth is 2 and the relative price is 1, solve his utility maximization problem if it is well defined.

   The problem is defined as $\max_{x\in\mathbb{R}^2_+} u(x,y)$ subject to $x + y \leq 2$ (assuming that the wealth of two is in relative terms). We can re-express this problem in such a way that all the constraints are explicit and therefore we have that we want to find (x,y) that solves the following problem $\max_{x\in\mathbb{R}^2} u(x,y)$ subject to $x+y \leq 2$, $x \geq 0$ and $y \geq 0$

   The Lagrangian is as follows:

$$L(x,\lambda) = -(x-4)^2 - y^2 - \lambda(x+y-2) + \mu_1 x + \mu_2 y.$$

$$\frac{\partial L}{\partial x} = -2(x-4) - \lambda + \mu_1 = 0, \quad \frac{\partial L}{\partial y} = 2y - \lambda + \mu_2 = 0 \tag{1}$$

$$\lambda(x+y-2) = 0, \quad \mu_1 x = 0, \quad \mu_2 y = 0 \tag{CS}$$

$$\lambda, \mu_1, \mu_2 \geq 0, \quad x+y \leq 2, \quad x, y \geq 0. \tag{2}$$

   Discussing by (CS) we have 8 cases.

   - **Case 1** $\lambda = \mu_1 = \mu_2 = 0$ Then by (1) we have that $y = 0$ and $x = 4$ which contradicts the constraint that $x + y \leq 2$
   - **Case 2** $\lambda \neq 0, \mu_1 = \mu_2 = 0$ Given that $\lambda \neq 0$ we must have that $x + y = 2$ (i). Given that $\mu_1 = \mu_2 = 0$ then by (1) we have that $\lambda = 2y = -2x + 8$, therefore $y = 4 - x$, plugging in (i) we have that $x + 4 - x = 2$ which is a contradiction
   - **Case 3** $\mu_1 \neq 0, \lambda = \mu_2 = 0$ Then by (CS) we have that $x = 0$. By (1) then we have that $y = 0$. But if $x = y = 0$ then by (1) we have that $\mu_1 = -8$ contradiction.
   - **Case 4** $\mu_2 \neq 0, \mu_1 = \lambda = 0$ Then by (CS) we have that $y = 0$. By (1) $\mu_2 = 0$ so we are back to case 1
   - **Case 5** $\lambda = 0, \mu_1 \neq 0, \mu_2 \neq 0$ Then by (CS) we must have that $x = y = 0$, but then from 1 we get that $\mu_2 = 0$, and $\mu_1 = -8$ which is a contradiction.
   - **Case 6** $\mu_1 = 0, \lambda \neq 0, \mu_2 \neq 0$ Then by (CS) we have that $y = 0$, and $x + y = 2$, therefore $x = 2$. By (1) we get that $\mu_2 = \lambda$, and $\lambda = 4$
   - **Case 7** $\mu_2 = 0, \mu_1 \neq 0, \lambda \neq 0$ Then by (CS) we have that $x = 0$ and $x + y = 2$, therefore $y = 2$. From the second equation in (1) we get that $\lambda = 4$, if so, from the first equation we get that $\mu_1 = -4$ which is a contradiction.

[1] If you find any typo please email me: Maria_Jose_Boccardi@Brown.edu

- **Case 8** $\lambda \neq 0, \mu_1 \neq 0, \mu_2 \neq 0$ Therefore by (CS) we must have that $x = 0$, $y = 0$ and $x + y = 2$ which is a contradiction.

Therefore, the unique solution is $(x^*, y^*, \lambda, \mu_1, \mu_2) = (2, 0, 4, 0, 4)$ and $u(x^*, y^*) = -4$.

2. Let $f : \mathbb{R}_+ \to \mathbb{R}$ and $f(x) = -(x+1)^2 + 2$. Solve the maximization problem if it is well defined.

The Lagrangian is as follows:

$$L(x, \lambda) = -(x+1)^2 + 2 + \lambda(x - 0).$$

$$\frac{\partial L}{\partial x} = -2(x+1) + \lambda = 0 \tag{3}$$

$$\lambda x = 0 \tag{CS}$$

$$\lambda \geq 0, \quad x \geq 0. \tag{4}$$

If $\lambda = 0$, $x = -1$ by (3), which contradicts (4). If $\lambda > 0$, $x = 0$ by (CS) and there is no contradiction. Since f is decreasing on the constraint set, 0 is the unique maximizer.

3. Let $f : \mathbb{R}^2_+ \to \mathbb{R}$ and $f(x, y) = 2y - x^2$. When (x, y) must be on the unit disc, i.e., $x^2 + y^2 \leq 1$, solve the *minimization* problem if it is well defined.[2]

The Lagrangian is as follows:

$$L(x, y, \lambda, \mu_1, \mu_2) = 2y - x^2 + \lambda(x^2 + y^2 - 1) - \mu_1 x - \mu_2 y.$$

$$\frac{\partial L}{\partial x} = -2x + 2\lambda x - \mu_1 = 0, \quad \frac{\partial L}{\partial y} = 2 + 2\lambda y - \mu_2 = 0 \tag{5}$$

$$\lambda(x^2 + y^2 - 1) = 0, \quad \mu_1 x = 0, \quad \mu_2 y = 0 \tag{CS}$$

$$\lambda, \mu_1, \mu_2 \geq 0, \quad x^2 + y^2 \leq 1, \quad x, y \geq 0. \tag{6}$$

If $\mu_1 \neq 0$, $x = 0$ by (CS). By (5), $\mu_1 = 0$, which is a contradiction. Thus $\mu_1 = 0$ (if a solution exists).

If $\lambda = 0$, $x = 0$ and $\mu_2 = 2$ by (5). By (CS), $y = 0$. This is a candidate of the solution.

If $\lambda \neq 0$, $x^2 + y^2 - 1 = 0$ by (CS). If $\mu_2 \neq 0$, $y = 0$ by (CS) and $\mu_2 = 2$ by (5). By (6), $x = 1$ and $\lambda = 1$ by (5). This is another candidate. If $\mu_2 = 0$, $\lambda y = -1$, which is a contradiction to (6).

Since $f(0, 0) = 0 > -1 = f(1, 0)$, the unique candidate is $(x^*, y^*, \lambda, \mu_1, \mu_2) = (1, 0, 1, 0, 2)$ and $f(x^*, y^*) = -1$.

If $\min f(x, y) < -1$, there exists $(\tilde{x}, \tilde{y})$ on the constraint set such that $2\tilde{y} - \tilde{x}^2 < -1$. By (6), we have $1 \leq 2\tilde{y} + 1 < \tilde{x}^2$, which implies that $|\tilde{x}| > 1$. This contradicts (6).

Therefore, the unique solution is $(x^*, y^*, \lambda, \mu_1, \mu_2) = (1, 0, 1, 0, 2)$ and $f(x^*, y^*) = -1$.

[2] This is the same problem as in Example 18.11 of Simon and Blume (1994).

2 Homework

1. **Exercise 18.3** Find the point on the parabola $y = x^2$ that is closest to the point $(2, 1)$. (Estimate the solution to the cubic equation which results)

   The problem then is to $\min_{x \in \mathbb{R}^{\not{z}}} d((x, y), (2, 1)) \sqrt{(x-2)^2 + (y-1)^2}$ subject to $x^2 - y = 0$. The easiest way to solve the problem is just to solve for y in the constraint and the plug it in the objective function, which now will be a function only on x and without any constraint. Therefore we have that $y = x^2$ and plugging into the objective function we get that we can rewrite the problem as $\min_{x \in \mathbb{R}^2} \sqrt{(x-2)^2 + (x^2-1)^2}$. Since the function $f(x) = \sqrt{x}$ is a monotonically increasing function we can further simplify the problem and rewrite it as $\min_{x \in \mathbb{R}^2} (x-2)^2 + (x^2-1)^2$ where the first order condition of the problem is given by

   $$x^3 - \frac{1}{2}x - 1 = 0$$

   . We know that for $x = 1.1$, $f(x) = x^3 - \frac{1}{2}x - 1 < 0$ and for $x = 1.2$, $f(x) = x^3 - \frac{1}{2}x - 1 > 0$, so x should be in $(1.1, 1.2)$. The actual solution is $x^* = 1.165$

2. **Exercise 18.6** Find the max and the min of $f(x, y, z) = x + y + z^2$ subject to $x^2 + y^2 + z^2 = 1$ and $y = 0$.

   The fastest way is to use the constraint $y = 0$ and simplify the problem to work with only two variables. Therefore we have that we can rewrite the problem as, find the max and the min of $f(x, y, z) = x + z^2$ subject to $x^2 + z^2 = 1$. The Lagrangian for this problem is given by

   $$L(x, z, \lambda) = x + z^2 - \lambda(x^2 + z^2 = 1)$$

   The NDCQ is given by

   $$\nabla h(x, z) = \begin{pmatrix} 2x \\ 2z \end{pmatrix}$$

   where the NDCQ is satisfied if it is not the case that $x = z = 0$, which we know cannot be the case in the optimum since $x^2 + z^2 = 1$. The FOC are given by

   $$2z - 2\lambda z = 0$$

   $$1 - 2\lambda x = 0$$

   therefore we have that $\lambda = 1$, $x = \frac{1}{2}$ and that $z^2 = \frac{3}{4}$ and therefore $z = \pm\frac{\sqrt{3}}{2}$. Therefore the solution candidates are given by $\left(\frac{1}{2}, 0, \frac{\sqrt{3}}{2}\right)$ and $\left(\frac{1}{2}, 0, -\frac{\sqrt{3}}{2}\right)$

3. **Exercise 18.7** Maximize $f(x, y, z) = yz + xz$ subject to $y^2 + z^2 = 1$ and $xz = 3$

   DONE in class

4. **Exercise 18.10** Find the maximizer of $f(x, y) = x^2 + y^2$, subject to the constraints $2x + y \leq 2$, $x \geq 0$ and $y \geq 0$

   The Lagrangian is as follows:

   $$L(x, \lambda) = x^2 + y^2 - \lambda(2x + y - 2) + \mu_1 x + \mu_2 y.$$

   $$\frac{\partial L}{\partial x} = 2x - 2\lambda + \mu_1 = 0, \quad \frac{\partial L}{\partial y} = 2y - \lambda + \mu_2 = 0 \tag{7}$$

   $$\lambda(2x + y - 2) = 0, \quad \mu_1 x = 0, \quad \mu_2 y = 0 \tag{CS}$$

   $$\lambda, \mu_1, \mu_2 \geq 0, \quad x + y \leq 2, \quad x, y \geq 0. \tag{8}$$

Discussing by (CS) we have 8 cases.

- **Case 1** $\lambda = \mu_1 = \mu_2 = 0$ Then by (1) we have that $x = 0$ and $y = 0$.
- **Case 2** $\lambda \neq 0, \mu_1 = \mu_2 = 0$ Given that $\lambda \neq 0$ we must have that $2x + y = 2$, therefore $y = 2 - 2x$ (i). Given that $\mu_1 = \mu_2 = 0$ then by (1) we have that $2x - 2\lambda = 0$ and $2(2 - 2x) - \lambda = 0$, therefore $\lambda = 4 - 4x = x$, then we have that $x = \frac{4}{5}$. Therefore we have that $y = \frac{2}{5}$ and $\lambda_1 = \frac{4}{5}$
- **Case 3** $\mu_1 \neq 0, \lambda = \mu_2 = 0$ Then by (CS) we have that $x = 0$. By (1) then we have that $\mu_1 = 0$ so we are back to case 1
- **Case 4** $\mu_2 \neq 0, \mu_1 = \lambda = 0$ Then by (CS) we have that $y = 0$. By (1) $\mu_2 = 0$ so we are back to case 1
- **Case 5** $\lambda = 0, \mu_1 \neq 0, \mu_2 \neq 0$ Then by (CS) we must have that $x = y = 0$, but then from 1 we get that $\mu_1 = 0$, and $\mu_2 = 0$ so we are back to case 1.
- **Case 6** $\mu_1 = 0, \lambda \neq 0, \mu_2 \neq 0$ Then by (CS) we have that $y = 0$, and $2x + y = 2$, therefore $x = 1$. By (1) we get that $2 - 2\lambda = 0$, therefore $\lambda = 1$, and we get that $\lambda = \mu_2$.
- **Case 7** $\mu_2 = 0, \mu_1 \neq 0, \lambda \neq 0$ Then by (CS) we have that $x = 0$ and $2x + y = 2$, therefore $y = 2$. From the second equation in (1) we get that $\lambda = 4$, and therefore from the first one we have that $\mu_1 = 8$.
- **Case 8** $\lambda \neq 0, \mu_1 \neq 0, \mu_2 \neq 0$ Therefore by (CS) we must have that $x = 0$, $y = 0$ and $2x + y = 2$ which is a contradiction.

Therefore, we have four candidates: $(\frac{4}{5}, \frac{2}{5})$, $(1, 0)$, $(0, 0)$ and $(0, 2)$. the unique solution is $(x^*, y^*, \lambda, \mu_1, \mu_2) = (2, 0, 4, 0, 4)$ and $u(x^*, y^*) = -4$.

5. **Exercise 18.11** Find the maximizer of $f(x, y) = 2y^2 - x$, subject to the constraints $x^2 + y^2 \leq 1$, $x \geq 0$ and $y \geq 0$

   DONE in previous part

6. **Exercise 18.12** Consider the problem of maximizing $f(x, y, z) = xyz + z$, subject to the constraints $x^2 + y^2 + z \leq 6$, $x \geq 0$, $y \geq 0$ and $z \geq 0$

   (a) Write out a complete set of first order conditions for this problem

   The Lagrangian is as follows:

$$L(x, \lambda) = xyz + z - \lambda(x^2 + y^2 + z - 6) + \mu_1 x + \mu_2 y + \mu_3 z.$$

$$\frac{\partial L}{\partial x} = yz - 2x\lambda + \mu_1 = 0, \quad \frac{\partial L}{\partial y} = xz - 2y\lambda + \mu_2 = 0, \quad \frac{\partial L}{\partial z} = xy + 1 - \lambda + \mu_3 = 0 \tag{9}$$

$$\lambda(x^2 + y^2 + z - 6) = 0, \quad \mu_1 x = 0, \quad \mu_2 y = 0, \quad \mu_3 z = 0 \tag{CS}$$

$$\lambda, \mu_1, \mu_2, \mu_3 \geq 0, \quad x^2 + y^2 + z \leq 2, \quad x, y, z \geq 0. \tag{10}$$

(b) Determine whether or not the constraint $x^2 + y^2 + z \leq 6$ is binding at any solution

Suppose it is not binding, then we have that $\lambda = 0$, and therefore we can rewrite the conditions as

$$yz + \mu_1 = 0, \quad xz + \mu_2 = 0, \quad xy + 1 + \mu_3 = 0 \tag{11}$$
$$\mu_1 x = 0, \quad \mu_2 y = 0, \quad \mu_3 z = 0 \tag{CS}$$
$$\lambda, \mu_1, \mu_2, \mu_3 \geq 0, \quad x, y, z \geq 0. \tag{12}$$

Given the CS condition we have the following cases

- **Case 1** $\mu 1 = \mu_2 = \mu_3 = 0$ Then by the first condition we have that $xy = -1$ which cannot be the case since $x \geq 0$ and $y \geq 0$
- **Case 2** $\mu_1 \neq 0, \mu_2 = \mu_3 = 0$ Then by the first condition we have that $xy = -1$ which cannot be the case since $x \geq 0$ and $y \geq 0$
- **Case 3** $\mu_2 \neq 0, \mu_1 = \mu_3 = 0$ Then by the first condition we have that $xy = -1$ which cannot be the case since $x \geq 0$ and $y \geq 0$
- **Case 4** $\mu_3 \neq 0, \mu_2 = \mu_1 = 0$ Then by the first condition we have that $xy = -1 - \mu_3$ which cannot be the case since $x \geq 0$, $y \geq 0$ and $\mu_3 > 0$
- **Case 5** $\mu_1 = 0, \mu_2 \neq 0, \mu_3 \neq 0$ Then by the first condition we have that $xy = -1 - \mu_3$ which cannot be the case since $x \geq 0$, $y \geq 0$ and $\mu_3 > 0$
- **Case 6** $\mu_2 = 0, \mu_1 \neq 0, \mu_3 \neq 0$ Then by the first condition we have that $xy = -1 - \mu_3$ which cannot be the case since $x \geq 0$, $y \geq 0$ and $\mu_3 > 0$
- **Case 7** $\mu_3 = 0, \mu_2 \neq 0, \mu_1 \neq 0$ hen by the first condition we have that $xy = -1$ which cannot be the case since $x \geq 0$ and $y \geq 0$
- **Case 8** $\mu_1 \neq 0, \mu_2 \neq 0, \mu_3 \neq 0$ Then by the first condition we have that $xy = -1 - \mu_3$ which cannot be the case since $x \geq 0$, $y \geq 0$ and $\mu_3 > 0$

(c) Find a solution of the first order conditions that includes $x = 0$

If $x = 0$ then we can rewrite the FOC as

$$yz + \mu_1 = 0, \quad -2y\lambda + \mu_2 = 0, \quad 1 - \lambda + \mu_3 = 0 \tag{13}$$
$$\lambda(y^2 + z - 6) = 0, \quad \mu_2 y = 0, \quad \mu_3 z = 0 \tag{CS}$$
$$\lambda, \mu_1, \mu_2, \mu_3 \geq 0, \quad x^2 + y^2 + z \leq 2, \quad y, z \geq 0. \tag{14}$$

From the first equation we have that

$$yz = 0 \ \ \& \ \ \mu_1 = 0, \quad 2y\lambda = \mu_2, \quad 1 - \lambda = \mu_3 = 0$$

Therefore we have a case that satisfies the FOC when $\mu_2 = \mu_3 = 0$, therefore $\lambda = 1$, $y = 0$ and $z = 6$

(d) Find three equations in the three unknowns x, y, z that must be satisfied if $x \neq 0$ at the solution

If we impose the condition that $x \neq 0 \Rightarrow x > 0$ we have that the FOC are given by

$$yz - 2x\lambda = 0, \quad xz - 2y\lambda + \mu_2 = 0, \quad xy + 1 - \lambda + \mu_3 = 0 \tag{15}$$
$$x^2 + y^2 + z - 6 = 0, \quad \mu_1 = 0, \quad \mu_2 y = 0, \quad \mu_3 z = 0 \tag{CS}$$
$$\lambda, \mu_2, \mu_3 \geq 0, \quad y, z \geq 0. \tag{16}$$

Therefore we have different cases depending on the values of μ_2 and μ_3 (we already prove that there is no solution when $\lambda = 0$)

- **Case 1** $\mu 2 = \mu_3 = 0$ Then we can rewrite the conditions as

$$yz - 2x\lambda = 0, \quad xz - 2y\lambda = 0, \qquad xy + 1 - \lambda = 0 \tag{17}$$
$$x^2 + y^2 + z - 6 = 0, \quad \mu_1 = 0, \tag{CS}$$
$$\lambda > 0, \qquad y, z \geq 0. \tag{18}$$

Therefore we can solve for λ and we get the condition that, given that $x, y \geq 0$ it should be the case that $x = y$, and therefore the conditions that the solution must satisfy in the optimum are

$$x^2 + y^2 + z = 6 \quad x = y \quad xy + 1 - \frac{z}{2} = 0$$

- **Case 2** $\mu_2 \neq 0, \mu_3 = 0$ Then we must have that $y = 0$ which contradicts the fact that $-2x\lambda = 0$, when we know that we don't have a solution if $\lambda = 0$
- **Case 3** $\mu_3 \neq 0, \mu_2 = 0$ Then we must have that $z = 0$ which contradicts the fact that $-2x\lambda = 0$, when we know that we don't have a solution if $\lambda = 0$
- **Case 4** $\mu_2 \neq 0, \mu_3 \neq 0$ Then we must have that $y = z = 0$ which contradicts the fact that $-2x\lambda = 0$, when we know that we don't have a solution if $\lambda = 0$

(e) Show that $x = 1$, $y = 1$ and $z = 4$ satisfies these equations