Constrained Optimization Solutions

Math Camp 2012

1 Exercises

1. There are two commodities: \( x \) and \( y \). Let the consumer’s consumption set be \( \mathbb{R}^2_+ \) and his preference relation on his consumption set be represented by \( u(x, y) = -(x - 4)^2 - y^2 \). When his initial wealth is 2 and the relative price is 1, solve his utility maximization problem if it is well defined.

The problem is defined as \( \max_{x \in \mathbb{R}^2_+} u(x, y) \) subject to \( x + y \leq 2 \) (assuming that the wealth of two is in relative terms). We can re-express this problem in such a way that all the constraints are explicit and therefore we have that we want to find \( (x, y) \) that solves the following problem

\[
\max_{x \in \mathbb{R}^2} u(x, y) \quad \text{subject to} \quad x + y \leq 2, \quad x \geq 0 \quad \text{and} \quad y \geq 0.
\]

The Lagrangian is as follows:

\[
L(x, \lambda) = -(x - 4)^2 - y^2 - \lambda(x + y - 2) + \mu_1 x + \mu_2 y.
\]

\[
\frac{\partial L}{\partial x} = -2(x - 4) - \lambda + \mu_1 = 0, \quad \frac{\partial L}{\partial y} = 2y - \lambda + \mu_2 = 0
\]

\[
\lambda(x + y - 2) = 0, \quad \mu_1 x = 0, \quad \mu_2 y = 0
\]  
\[
\lambda, \mu_1, \mu_2 \geq 0, \quad x + y \leq 2, \quad x, y \geq 0.
\]

Discussing by (CS) we have 8 cases.

- **Case 1** \( \lambda = \mu_1 = \mu_2 = 0 \) Then by (1) we have that \( y = 0 \) and \( x = 4 \) which contradicts the constraint that \( x + y \leq 2 \).

- **Case 2** \( \lambda \neq 0, \mu_1 = \mu_2 = 0 \) Given that \( \lambda \neq 0 \) we must have that \( x + y = 2 \) (i). Given that \( \mu_1 = \mu_2 = 0 \) then by (1) we have that \( \lambda = 2y = -2x + 8 \), therefore \( y = 4 - x \), plugging in (i) we have that \( x + 4 - x = 2 \) which is a contradiction

- **Case 3** \( \mu_1 \neq 0, \lambda = \mu_2 = 0 \) Then by (CS) we have that \( x = 0 \). By (1) then we have that \( y = 0 \). But if \( x = y = 0 \) then by (1) we have that \( \mu_1 = -8 \) contradiction.

- **Case 4** \( \mu_2 \neq 0, \mu_1 = \lambda = 0 \) Then by (CS) we have that \( y = 0 \). By (1) \( \mu_2 = 0 \) so we are back to case 1

- **Case 5** \( \lambda = 0, \mu_1 \neq 0, \mu_2 \neq 0 \) Then by (CS) we must have that \( x = y = 0 \), but then from 1 we get that \( \mu_2 = 0 \), and \( \mu_1 = -8 \) which is a contradiction.

- **Case 6** \( \mu_1 = 0, \lambda \neq 0, \mu_2 \neq 0 \) Then by (CS) we have that \( y = 0 \), and \( x + y = 2 \), therefore \( x = 2 \). By (1) we get that \( \mu_2 = \lambda \), and \( \lambda = 4 \)

- **Case 7** \( \mu_2 = 0, \mu_1 \neq 0, \lambda \neq 0 \) Then by (CS) we have that \( x = 0 \) and \( x + y = 2 \), therefore \( y = 2 \). From the second equation in (1) we get that \( \lambda = 4 \), if so, from the first equation we get that \( \mu_1 = -4 \) which is a contradiction.
2 Constrained Optimization Solutions

• **Case 8** \( \lambda \neq 0, \mu_1 \neq 0, \mu_2 \neq 0 \) Therefore by (CS) we must have that \( x = 0, y = 0 \) and \( x + y = 2 \) which is a contradiction.

Therefore, the unique solution is \((x^*, y^*, \lambda, \mu_1, \mu_2) = (2, 0, 4, 0, 4)\) and \( u(x^*, y^*) = -4 \).

2. Let \( f : \mathbb{R}_+ \rightarrow \mathbb{R} \) and \( f(x) = -(x+1)^2 + 2 \). Solve the maximization problem if it is well defined.

The Lagrangian is as follows:

\[
L(x, \lambda) = -(x+1)^2 + 2 + \lambda(x - 0).
\]

\[
\frac{\partial L}{\partial x} = -2(x+1) + \lambda = 0 \tag{3}
\]

\[
\lambda x = 0 \tag{CS}
\]

\[
\lambda \geq 0, \quad x \geq 0. \tag{4}
\]

If \( \lambda = 0, \) \( x = -1 \) by (3), which contradicts (4). If \( \lambda > 0, \) \( x = 0 \) by (CS) and there is no contradiction. Since \( f \) is decreasing on the constraint set, \( 0 \) is the unique maximizer.

3. Let \( f : \mathbb{R}_+^2 \rightarrow \mathbb{R} \) and \( f(x, y) = 2y - x^2 \). When \((x, y)\) must be on the unit disc, i.e., \( x^2 + y^2 \leq 1 \), solve the minimization problem if it is well defined.

The Lagrangian is as follows:

\[
L(x, y, \lambda, \mu_1, \mu_2) = 2y - x^2 + \lambda(x^2 + y^2 - 1) - \mu_1 x - \mu_2 y.
\]

\[
\frac{\partial L}{\partial x} = -2x + 2\lambda x - \mu_1 = 0,
\]

\[
\frac{\partial L}{\partial y} = 2 + 2\lambda y - \mu_2 = 0 \tag{5}
\]

\[
\lambda(x^2 + y^2 - 1) = 0, \quad \mu_1 x = 0, \quad \mu_2 y = 0 \tag{CS}
\]

\[
\lambda, \mu_1, \mu_2 \geq 0, \quad x^2 + y^2 \leq 1, \quad x, y \geq 0. \tag{6}
\]

If \( \mu_1 \neq 0, \) \( x = 0 \) by (CS). By (5), \( \mu_1 = 0 \), which is a contradiction. Thus \( \mu_1 = 0 \) (if a solution exists).

If \( \lambda = 0, \) \( x = 0 \) and \( \mu_2 = 2 \) by (5). By (CS), \( y = 0 \). This is a candidate of the solution.

If \( \lambda \neq 0, \) \( x^2 + y^2 - 1 = 0 \) by (CS). If \( \mu_2 \neq 0, \) \( y = 0 \) by (CS) and \( \mu_2 = 2 \) by (5). By (6), \( x = 1 \) and \( \lambda = 1 \) by (5). This is another candidate. If \( \mu_2 = 0, \) \( \lambda y = -1 \), which is a contradiction to (6).

Since \( f(0, 0) = 0 > -1 = f(1, 0) \), the unique candidate is \((x^*, y^*, \lambda, \mu_1, \mu_2) = (1, 0, 1, 0, 2)\) and \( f(x^*, y^*) = -1 \).

If \( \min f(x, y) < -1 \), there exists \((\tilde{x}, \tilde{y})\) on the constraint set such that \( 2\tilde{y} - \tilde{x}^2 < -1 \). By (6), we have \( 1 \leq 2\tilde{y} + 1 < \tilde{x}^2 \), which implies that \( |\tilde{x}| > 1 \). This contradicts (6).

Therefore, the unique solution is \((x^*, y^*, \lambda, \mu_1, \mu_2) = (1, 0, 1, 0, 2)\) and \( f(x^*, y^*) = -1 \).

---

\[\text{This is the same problem as in Example 18.11 of Simon and Blume (1994).}\]
2 Homework

1. **Exercise 18.3** Find the point on the parabola \( y = x^2 \) that is closest to the point \((2,1)\). (Estimate the solution to the cubic equation which results)

   The problem then is to \( \min_{x \in \mathbb{R}} d((x, y), (2, 1)) \sqrt{(x - 2)^2 + (y - 1)^2} \) subject to \( x^2 - y = 0 \).
   
   The easiest way to solve the problem is just to solve for \( y \) in the constraint and plug it into the objective function, which now will be a function only on \( x \) and without any constraint. Therefore we have that \( y = x^2 \) and plugging into the objective function we get that we can rewrite the problem as \( \min_{x \in \mathbb{R}} (x - 2)^2 + (x^2 - 1)^2 \). Since the function \( f(x) = \sqrt{x} \) is a monotonically increasing function we can further simplify the problem and rewrite it as \( \min_{x \in \mathbb{R}} (x - 2)^2 + (x^2 - 1)^2 \) where the first order condition of the problem is given by
   
   \[
   x^3 - \frac{1}{2} x - 1 = 0
   \]

   We know that for \( x = 1.1 \), \( f(x) = x^3 - \frac{1}{2} x - 1 < 0 \) and for \( x = 1.2 \), \( f(x) = x^3 - \frac{1}{2} x - 1 > 0 \), so \( x \) should be in \((1.1, 1.2)\). The actual solution is \( x^* = 1.165 \).

2. **Exercise 18.6** Find the max and the min of \( f(x, y, z) = x + y + z^2 \) subject to \( x^2 + y^2 + z^2 = 1 \) and \( y = 0 \).

   The fastest way is to use the constraint \( y = 0 \) and simplify the problem to work with only two variables. Therefore we have that we can rewrite the problem as, find the max and the min of \( f(x, y, z) = x + z^2 \) subject to \( x^2 + z^2 = 1 \). The Lagrangian for this problem is given by
   
   \[
   L(x, z, \lambda) = x + z^2 - \lambda(x^2 + z^2 = 1)
   \]

   The NDCQ is given by
   
   \[
   \nabla h(x, z) = \left( \begin{array}{c} 2x \\ 2z \end{array} \right)
   \]

   where the NDCQ is satisfied if it is not the case that \( x = z = 0 \), which we know cannot be the case in the optimum since \( x^2 + z^2 = 1 \). The FOC are given by
   
   \[
   2z - 2\lambda z = 0 \\
   1 - 2\lambda x = 0
   \]

   therefore we have that \( \lambda = 1 \), \( x = \frac{1}{2} \) and that \( z^2 = \frac{3}{4} \) and therefore \( z = \pm \frac{\sqrt{3}}{2} \). Therefore the solution candidates are given by \((\frac{1}{2}, 0, \frac{\sqrt{3}}{2})\) and \((\frac{1}{2}, 0, -\frac{\sqrt{3}}{2})\).

3. **Exercise 18.7** Maximize \( f(x, y, z) = yz + xz \) subject to \( y^2 + z^2 = 1 \) and \( xz = 3 \)

   DONE in class

4. **Exercise 18.10** Find the maximizer of \( f(x, y) = x^2 + y^2 \), subject to the constraints \( 2x + y \leq 2 \), \( x \geq 0 \) and \( y \geq 0 \).

   The Lagrangian is as follows:
   
   \[
   L(x, \lambda) = x^2 + y^2 - \lambda(2x + y - 2) + \mu_1 x + \mu_2 y.
   \]

   \[
   \frac{\partial L}{\partial x} = 2x - 2\lambda + \mu_1 = 0, \quad \frac{\partial L}{\partial y} = 2y - \lambda + \mu_2 = 0 \tag{7}
   \]

   \[
   \lambda(2x + y - 2) = 0, \quad \mu_1 x = 0, \quad \mu_2 y = 0 \tag{CS}
   \]

   \[
   \lambda, \mu_1, \mu_2 \geq 0, \quad x + y \leq 2, \quad x, y \geq 0. \tag{8}
   \]
Discussing by (CS) we have 8 cases.

- **Case 1** $\lambda = \mu_1 = \mu_2 = 0$ Then by (1) we have that $x = 0$ and $y = 0$.

- **Case 2** $\lambda \neq 0, \mu_1 = \mu_2 = 0$ Given that $\lambda \neq 0$ we must have that $2x + y = 2$, therefore $y = 2 - 2x$ (i). Given that $\mu_1 = \mu_2 = 0$ then by (1) we have that $2x - 2\lambda = 0$ and $2(2 - 2x) - \lambda = 0$, therefore $\lambda = 4 - 4x = x$, then we have that $x = \frac{4}{5}$. Therefore we have that $y = \frac{2}{5}$ and $\lambda_1 = \frac{4}{5}$.

- **Case 3** $\mu_1 \neq 0, \lambda = \mu_2 = 0$ Then by (CS) we have that $x = 0$. By (1) then we have that $\mu_1 = 0$ so we are back to case 1.

- **Case 4** $\mu_2 \neq 0, \mu_1 = \lambda = 0$ Then by (CS) we have that $y = 0$. By (1) $\mu_2 = 0$ so we are back to case 1.

- **Case 5** $\lambda = 0, \mu_1 \neq 0, \mu_2 \neq 0$ Then by (CS) we must have that $x = y = 0$, but then from 1 we get that $\mu_1 = 0$, and $\mu_2 = 0$ so we are back to case 1.

- **Case 6** $\mu_1 = 0, \lambda \neq 0, \mu_2 \neq 0$ Then by (CS) we have that $y = 0$, and $2x + y = 2$, therefore $x = 1$. By (1) we get that $2 - 2\lambda = 0$, therefore $\lambda = 1$, and we get that $\lambda = \mu_2$.

- **Case 7** $\mu_2 = 0, \mu_1 \neq 0, \lambda \neq 0$ Then by (CS) we have that $x = 0$ and $2x + y = 2$, therefore $y = 2$. From the second equation in (1) we get that $\lambda = 4$, and therefore from the first one we have that $\mu_1 = 8$.

- **Case 8** $\lambda \neq 0, \mu_1 \neq 0, \mu_2 \neq 0$ Therefore by (CS) we must have that $x = 0$, $y = 0$ and $2x + y = 2$ which is a contradiction.

Therefore, we have four candidates: $(\frac{4}{5}, \frac{2}{5}), (1,0), (0,0)$ and $(0,2)$. The unique solution is $(x^*, y^*, \lambda, \mu_1, \mu_2) = (2,0,4,0,4)$ and $u(x^*, y^*) = -4$.

5. **Exercise 18.11** Find the maximizer of $f(x, y) = 2y^2 - x$, subject to the constraints $x^2 + y^2 \leq 1$, $x \geq 0$ and $y \geq 0$.

DONE in previous part.

6. **Exercise 18.12** Consider the problem of maximizing $f(x, y, z) = xyz + z$, subject to the constraints $x^2 + y^2 + z \leq 6$, $x \geq 0$, $y \geq 0$ and $z \geq 0$.

(a) Write out a complete set of first order conditions for this problem.

The Lagrangian is as follows:

$$L(x, \lambda) = xyz + z - \lambda(x^2 + y^2 + z - 6) + \mu_1 x + \mu_2 y + \mu_3 z.$$  

$$\frac{\partial L}{\partial x} = yz - 2x\lambda + \mu_1 = 0, \quad \frac{\partial L}{\partial y} = xz - 2y\lambda + \mu_2 = 0, \quad \frac{\partial L}{\partial z} = xy + 1 - \lambda + \mu_3 = 0$$

$$\lambda(x^2 + y^2 + z - 6) = 0, \quad \mu_1 x = 0, \quad \mu_2 y = 0, \quad \mu_3 z = 0$$

$$\lambda, \mu_1, \mu_2, \mu_3 \geq 0, \quad x^2 + y^2 + z \leq 2, \quad x, y, z \geq 0.$$  

(CS)  

(10)
(b) Determine whether or not the constraint \(x^2 + y^2 + z \leq 6\) is binding at any solution

Suppose it is not binding, then we have that \(\lambda = 0\), and therefore we can rewrite the conditions as

\[
\begin{align*}
    yz + \mu_1 &= 0, \quad xz + \mu_2 = 0, \quad xy + 1 + \mu_3 = 0 \\
    \mu_1 x &= 0, \quad \mu_2 y = 0, \quad \mu_3 z = 0 \\
    \lambda, \mu_1, \mu_2, \mu_3 &\geq 0, \quad x, y, z \geq 0. 
\end{align*}
\]

(Given the CS condition we have the following cases)

- **Case 1** \(\mu_1 = \mu_2 = \mu_3 = 0\) Then by the first condition we have that \(xy = -1\) which cannot be the case since \(x \geq 0\) and \(y \geq 0\)
- **Case 2** \(\mu_1 \neq 0, \mu_2 = \mu_3 = 0\) Then by the first condition we have that \(xy = -1\) which cannot be the case since \(x \geq 0\) and \(y \geq 0\)
- **Case 3** \(\mu_2 = 0, \mu_1 = \mu_3 = 0\) Then by the first condition we have that \(xy = -1\) which cannot be the case since \(x \geq 0\) and \(y \geq 0\)
- **Case 4** \(\mu_3 \neq 0, \mu_2 = \mu_1 = 0\) Then by the first condition we have that \(xy = -1 - \mu_3\) which cannot be the case since \(x \geq 0\), \(y \geq 0\) and \(\mu_3 > 0\)
- **Case 5** \(\mu_1 = 0, \mu_2 \neq 0, \mu_3 = 0\) Then by the first condition we have that \(xy = -1 - \mu_3\) which cannot be the case since \(x \geq 0\), \(y \geq 0\) and \(\mu_3 = 0\)
- **Case 6** \(\mu_2 = 0, \mu_1 \neq 0, \mu_3 \neq 0\) Then by the first condition we have that \(xy = -1 - \mu_3\) which cannot be the case since \(x \geq 0\), \(y \geq 0\) and \(\mu_3 > 0\)
- **Case 7** \(\mu_3 = 0, \mu_2 \neq 0, \mu_3 \neq 0\) Then by the first condition we have that \(xy = -1\) which cannot be the case since \(x \geq 0\), \(y \geq 0\) and \(\mu_3 > 0\)
- **Case 8** \(\mu_1 \neq 0, \mu_2 \neq 0, \mu_3 \neq 0\) Then by the first condition we have that \(xy = -1 - \mu_3\) which cannot be the case since \(x \geq 0\), \(y \geq 0\) and \(\mu_3 > 0\)

(c) Find a solution of the first order conditions that includes \(x = 0\)

If \(x = 0\) then we can rewrite the FOC as

\[
\begin{align*}
    yz + \mu_1 &= 0, \quad -2y\lambda + \mu_2 = 0, \quad 1 - \lambda + \mu_3 = 0 \\
    \lambda(y^2 + z - 6) &= 0, \quad \mu_2 y = 0, \quad \mu_3 z = 0 \\
    \lambda, \mu_1, \mu_2, \mu_3 &\geq 0, \quad x^2 + y^2 + z \leq 2, \quad y, z \geq 0. 
\end{align*}
\]

From the first equation we have that

\[yz = 0 \quad \& \quad \mu_1 = 0, \quad 2y\lambda = \mu_2, \quad 1 - \lambda = \mu_3 = 0\]

Therefore we have a case that satisfies the FOC when \(\mu_2 = \mu_3 = 0\), therefore \(\lambda = 1, y = 0\) and \(z = 6\)

(d) Find three equations in the three unknowns \(x, y, z\) that must be satisfied if \(x \neq 0\) at the solution

If we impose the condition that \(x \neq 0 \Rightarrow x > 0\) we have that the FOC are given by

\[
\begin{align*}
    yz - 2x\lambda &= 0, \quad xz - 2y\lambda + \mu_2 = 0, \quad xy + 1 - \lambda + \mu_3 = 0 \\
    x^2 + y^2 + z - 6 &= 0, \quad \mu_1 = 0, \quad \mu_2 y = 0, \quad \mu_3 z = 0 \\
    \lambda, \mu_2, \mu_3 &\geq 0, \quad y, z \geq 0. 
\end{align*}
\]

Therefore we have different cases depending on the values of \(\mu_2\) and \(\mu_3\) (we already prove that there is no solution when \(\lambda = 0\))
• **Case 1** $\mu_2 = \mu_3 = 0$ Then we can rewrite the conditions as

\[
\begin{align*}
yz - 2x\lambda &= 0, \quad xz - 2y\lambda = 0, \quad xy + 1 - \lambda = 0 \quad (17) \\
x^2 + y^2 + z - 6 &= 0, \quad \mu_1 = 0, \quad (CS) \\
\lambda > 0, \quad y, z &\geq 0. \quad (18)
\end{align*}
\]

Therefore we can solve for $\lambda$ and we get the condition that, given that $x, y \geq 0$ it should be the case that $x = y$, and therefore the conditions that the solution must satisfy in the optimum are

\[
x^2 + y^2 + z = 6 \quad x = y \quad xy + 1 - \frac{z}{2} = 0
\]

• **Case 2** $\mu_2 \neq 0, \mu_3 = 0$ Then we must have that $y = 0$ which contradicts the fact that $-2x\lambda = 0$, when we know that we don’t have a solution if $\lambda = 0$

• **Case 3** $\mu_3 \neq 0, \mu_2 = 0$ Then we must have that $z = 0$ which contradicts the fact that $-2x\lambda = 0$, when we know that we don’t have a solution if $\lambda = 0$

• **Case 4** $\mu_2 \neq 0, \mu_3 \neq 0$ Then we must have that $y = z = 0$ which contradicts the fact that $-2x\lambda = 0$, when we know that we don’t have a solution if $\lambda = 0$

(e) Show that $x = 1, y = 1$ and $z = 4$ satisfies these equations