THE ENVELOPE THEOREM

Econ 2010 TA Section: María José Boccardi

Theorem 1 *(ET) Let $f:\mathbb{R}^L \to \mathbb{R}$ and $h:\mathbb{R}^{L+1} \to \mathbb{R}$ be C^1 functions. Suppose that the constrained optimization problem*

$$\begin{array}{ll} \max_{x\in\mathbb{R}^L} & f(x) \\ s.t. & h(x,a)=0 \end{array}$$

admits a unique solution $x^*(a)$ with an associated Lagrange multiplier $\mu^*(a)$. If $L(x,\mu,a)$ denotes the Lagrangean then

$$\frac{d}{da}f\left(x^*\left(a\right)\right) = \left.\frac{\partial L}{\partial a}\right|_{(x^*(a),\mu^*(a),a)}$$

Proof. Let define the Lagrangean as $L(x,\mu,a)=f(x)-\mu h(x,a)$

Where the first order conditions are given by

$$\begin{aligned} \frac{\partial L(x,\mu,a)}{\partial x_l} &= 0 \Leftrightarrow \\ \frac{\partial}{\partial x_l}f(x^*(a))-\mu^*(a)\frac{\partial}{\partial x_l}h(x^*(a),a) &= 0 \end{aligned}$$

for all $l\in\{1,...,L\}$; and

$$h(x^*(a),a)=0$$

From the FOC we can obtain

$$\begin{aligned} 0 &= \frac{d}{da}h\left(x^*\left(a\right),a\right) \\ &= \sum_{l=1}^{L}\frac{\partial}{\partial x_l}h(x^*(a),a)x_l^{'}(a)+\frac{\partial}{\partial a}h\left(x^*\left(a\right),a\right) \\ &= \sum_{l=1}^{L}\frac{\partial}{\partial x_l}f(x^*(a))\cdot\frac{x_l^{'}(a)}{\mu^*(a)}+\frac{\partial}{\partial a}h\left(x^*\left(a\right),a\right) \end{aligned}$$

Therefore

$$\sum_{l=1}^{L}\frac{\partial}{\partial x_l}f(x^*(a))x_l^{'}(a)=-\mu^*(a)\frac{\partial}{\partial a}h\left(x^*\left(a\right),a\right)$$

From this inequality we have that

$$\begin{aligned} \frac{d}{da}f\left(x^*\left(a\right)\right) &= \sum_{l=1}^{L}\frac{\partial}{\partial x_l}f(x^*(a))\cdot x_l^{'}(a) \\ &= -\mu^*(a)\frac{\partial}{\partial a}h\left(x^*\left(a\right),a\right) \\ &= \frac{\partial}{\partial a}L\left(x^*(a),\mu^*(a),a\right) \end{aligned}$$

■

Example 2 *Suppose that an agent faces a utility maximization problem*

$$\max_{x \in \mathbb{R}^L} U(x)$$

$$s.t \quad p^T x - I = 0$$

where $U \in C^1$ admits a unique solution. The envelope theorem shows that

$$\frac{d}{dI} U(x^*(I)) = \mu^*(I)$$

where $\mu^*(I)$ is the Lagrange multiplier. There is where it comes from the interpretation of the Lagrange Multipliers as the Shadow Value of Income.

Remark 3 *Other applications include :*

- – Policy analysis in general as, for example,effects of wealth redistribution on welfare.
  – Microeconomic Theory
    - ∗ Hicksian demand and Expenditure function (MWG p. 69)
    - ∗ Roy's Identity (MWG p.74)
    - ∗ Shepard's Lemma (MWG p.141)
    - ∗ Hotelling's Lemma (MWG p. 138)