# G5212: Game Theory

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# What is Missing?

- So far we have formally covered
  - Static Games of Complete Information
  - Dynamic Games of Complete Information
  - Static Games of Incomplete Information
- We have so far not covered Dynamic Games of Incomplete Information
  - Though we have studied games of **imperfect** information
- These are going to be crucial for our study of information economics
  - Will cover them more formally now

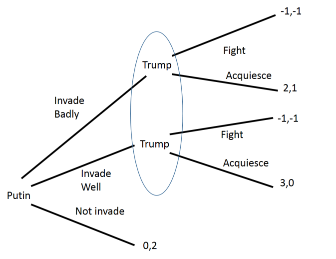
# Types of Incomplete Information

- The structure we are going to introduce will allow us to deal with two types of lack of information
  - About the actions that other players have previously taken (imperfect information)
  - About the state of the world (incomplete information)
- When thinking about static games we distinguished between these two types of uncertainty
  - Why?
- The latter type of game required us to think about Bayesian updating, while the former did not
- Here, both types on uncertainty mean that we may have to update beliefs
- We will deal with them in the same framework
  - Games in which the only uncertainty is about the state of the world are sometimes called *Bayesian games with observable actions*

# A Motivating Example

## Example

A Motivating Example

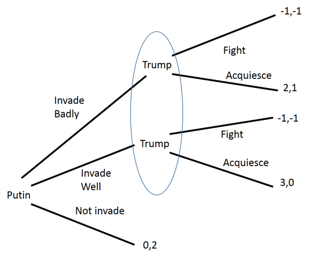



- What are the NE of this game?
- What are the SPNE of this game?

# A Motivating Example

## Example

A Motivating Example

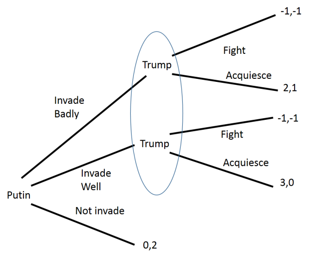



| | Fight | Not Fight |
|---|---|---|
| Invade Badly | $-1, -1$ | $2, 1$ |
| Invade Well | $-1, -1$ | $3, 0$ |
| Not Invade | $0, 2$ | $0, 2$ |

# A Motivating Example

## Example

A Motivating Example




- Two NE: (Not Invade, Fight) and (Invade Well, Not Fight)
- Only one of these is 'plausible'
- Let's use SPNE to rule the other one out
- uh oh.......

# Weak Perfect Bayesian Equilibrium

- How can we deal with this situation?
- One clue is in the fact that 'Fight' is dominated for Trump, **regardless of his beliefs**
- So perhaps we could do something about that?

**Definition**

A **system of beliefs** $\mu$ for an extensive form game is, for each information set $H$, a probability distribution $\mu_H$ over the decision nodes in that information set

- So, in the example above a system of beliefs assigns a probability to "Invade well" and "Invade badly"

# Weak Perfect Bayesian Equilibrium

- The first thing we could do is demand that players have beliefs, and best respond to those beliefs
- This is extending the notion of sequential rationality to this type of game

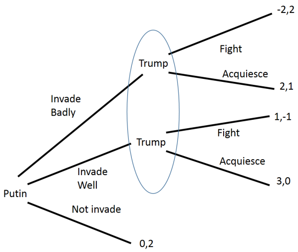
**Definition**

A strategy profile $(\sigma_1, ...\sigma_N)$ is sequentially rational at information set $H$ given beliefs $\mu$ if, for the player $i$ moving at $H$, the expected utility of $\sigma_i$ conditional on $\sigma_{-i}$ and $\mu_H$ is maximal over all possible strategies
A strategy profiles is sequentially rational given $\mu$ if it is sequentially rational at every information set

# Weak Perfect Bayesian Equilibrium

- In fact, in order to solve our previous problem, all we need is to require that players are sequentially rational given *some* system of beliefs
  - Whatever beliefs Trump has, it rules out playing Fight
  - Any equilibrium involves playing 'Not Fight'
- However, we probably want to assume more
  - At the moment what we have done is something similar to Rationalizability
  - We have done nothing to link beliefs to strategies

# Weak Perfect Bayesian Equilibrium




- In our definition so far it would be fine for Trump to fight, on the basis that he believes that Putin has chosen 'Invade Badly'
- But this is dominated strategy for Putin
- We probably want to rule this out

# Weak Perfect Bayesian Equilibrium

- In order to have a solution concept that is similar to Nash equilibrium, we add one further requirement
- The system of beliefs $\mu$ is derived from the strategy profile $\sigma$ using Bayes rule **wherever possible**
- i.e., assuming that information set $H$ is reached with positive probability given $\sigma$ it must be the case that for each node $x \in H$

$$\mu_H(x) = \frac{P(x|\sigma)}{P(H|\sigma)}$$

- Notice that 'wherever possible' is a rather large caveat here
- Generally speaking some nodes will **not** be reached with positive probability
- This will cause a host of problems (as we shall see)

# Weak Perfect Bayesian Equilibrium

**Definition**

A strategy profile $\sigma$ and a system of beliefs $\mu$ form a Weak Perfect Bayesian Equilibrium of an extensive game $\Gamma_E$ if

1. $\sigma$ is sequentially rational given $\mu$
2. $\mu$ is derived from $\sigma$ wherever possible

# Incorporating Types

- So far we have focussed only on a game in which all uncertainty is driven by the fact that Trump didn't observe an action of Putin
- What about uncertainty about the type of game?
  - What we have previously called games on Incomplete Information
- Here it can be convenient to use a 'trick'
  - The Harsanyi Transformation
- We introduce 'nature' as a player in the game
  - Always moves first
  - Doesn't have any payoffs
  - Plays according to prescribed probabilities
- Changes a game of incomplete information into a game of imperfect information

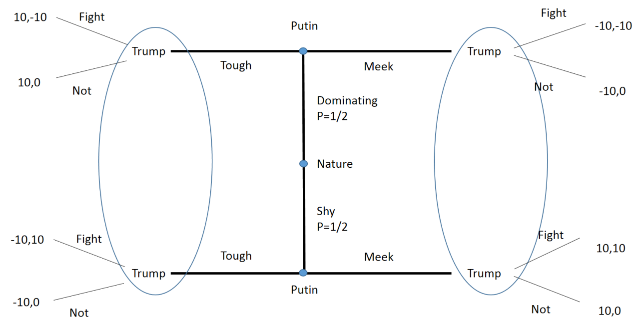
# Incorporating Types

## Example

Putin can be one of two types: dominating or shy, with equal probability. Conditional on his type he can choose one of two postures: Tough or Meek. Trump gets to observe the posture that Putin has taken and decide whether to fight or not. A shy Putin hates being Tough, and gets a payoff of -10 regardless of whether Trump fights or not. A dominating Putin loves being Tough, and gets a payoff of 10 regardless of what Trump does. If Trump does not fight he gets a payoff of zero. If he fights, he will beat a shy Putin (and get a payoff of 10) but lose to a dominating Putin (and get a payoff of -10)

# Incorporating Types

## Example




- Nature moves first
- Putin observes Nature's move and chooses a strategy
- Trump observes the strategy of Putin but not Nature's move

# Incorporating Types

## Example

- How can we find the WPB Equilibrium of this game?
- One way is as follows:
  - Note that T(ough) is a dominant strategy for a D(ominant) Putin
  - M(eek) is a dominant strategy for a S(hy) Putin
  - Thus, in any equilibrium, $\sigma_P(T|D) = \sigma_P(M|S) = 1$
  - Trump can therefore infer Putin's type from his action

$$
\begin{aligned}
\mu_T(D) &= P(D|T) = \frac{P(D \cap T)}{P(T)} = \frac{0.5}{0.5} = 1 \\
\mu_M(S) &= P(S|M) = \frac{P(S \cap M)}{P(M)} = \frac{0.5}{0.5} = 1
\end{aligned}
$$

# Incorporating Types

## Example

- Trump's best action is to fight if his belief that Putin is shy is greater than 0.5

$$\begin{aligned} u(F|\mu) &= (1-\mu(S))(-10)+\mu(S)10 \\ &= -10+20\mu(S) \\ &\geq u(N|\mu)=0 \\ &\Rightarrow \mu(S)\geq \frac{1}{2} \end{aligned}$$

- So it is optimal for Trump to fight if he sees Putin be meek, and not if he is being tough

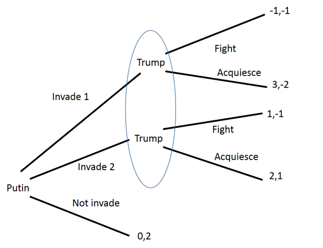
# Incorporating Types

## Example

- So a WPB Equilibrium of this game consists of
  - A strategy for Putin: $\sigma_P(T|D) = \sigma_P(M|S) = 1$
  - A strategy for Trump: $\sigma_T(N|T) = \sigma_T(F|M) = 1$
  - Beliefs: $\mu_T(D) = \mu_M(S) = 1$
- It should be fairly clear that this is a very simplistic example
  - We have turned off the channel by which a shy Putin might want to pretend to be Dominant
  - Clearly a lot of the interest in situations such as this comes from that tension
  - Don't worry - we will spend plenty of time looking at this channel!

# WPBE - Some More Practice

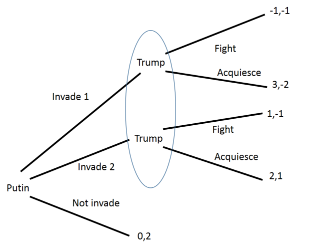
## Example




- Let's figure out the WPB equilibrium of this game
- First, when will Trump play Fight?

# WPBE - Some More Practice

## Example

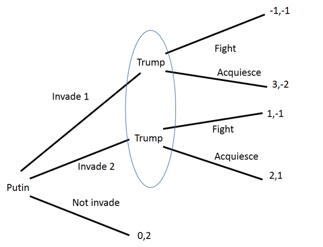



$$
\begin{aligned}
u(F|\mu) &= -1 \\
u(N|\mu) &= -2\mu(1) + (1 - \mu(1))
\end{aligned}
$$

- Will fight with positive probability only if $\mu(1) \geq \frac{2}{3}$

# WPBE - Some More Practice

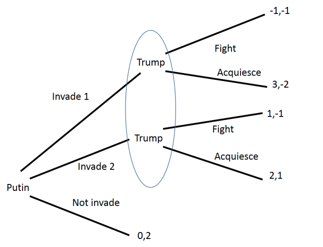
## Example




- Can there be an equilibrium where $\mu(1) > \frac{2}{3}$?
- No!
  - In this case Trump will play Fight
  - Putin plays Invade 2
  - Implies $\mu(1) = 0$

# WPBE - Some More Practice

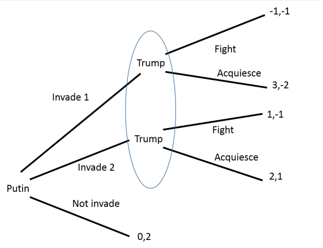
## Example




- Can there be an equilibrium where $\mu(1) < \frac{2}{3}$?
- No!
  - In this case Trump will play Not Fight
  - Putin plays Invade 1
  - Implies $\mu(1) = 1$

# WPBE - Some More Practice

## Example

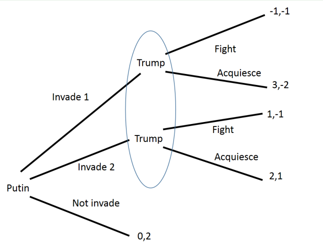



- Only possible WPBE is with $\mu(1) = \frac{2}{3}$
- Means Putin must be indifferent between Invade 1 and Invade 2

$$\begin{aligned} -\sigma_T(F) + 3(1 - \sigma_T(F)) &= 1\sigma_T(F) + 2(1 - \sigma_T(F)) \\ \Rightarrow \quad & \sigma_T(F) = \frac{1}{3} \end{aligned}$$

# WPBE - Some More Practice

## Example

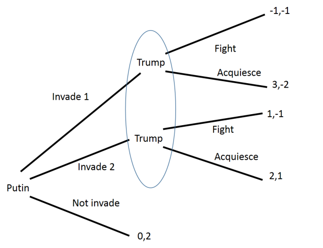



- Given this strategy of Trump

$$
\begin{aligned}
u_P(1, \sigma_T) &= \frac{-1}{3} + 3.\frac{2}{3} = \frac{5}{3} \\
u_P(2, \sigma_T) &= \frac{1}{3} + 2.\frac{2}{3} = \frac{5}{3} \\
u_P(Not\ Invade, \sigma_T) &= 0
\end{aligned}
$$

# WPBE - Some More Practice

## Example



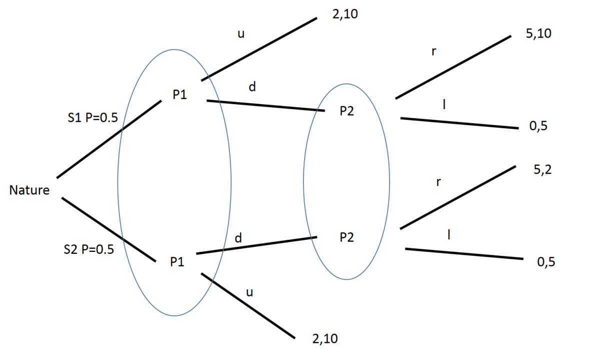

- Thus WPB Equilibrium of this game is
  - A strategy for Putin: $\sigma_P(1) = \frac{2}{3}$, $\sigma_P(2) = \frac{1}{3}$
  - A strategy for Trump: $\sigma_T(F|Invade) = \frac{1}{3}$, $\sigma_T(N|Invade) = \frac{2}{3}$
  - Beliefs: $\mu_T(1) = \frac{2}{3}$

# Problems with Weak Perfect Bayesian Equilibrium

- Given the way that this course has gone, you will be unsurprised that there are problems with WPB Equilibrium
- Most of these are to do with updating in the face of zero probability events
- Here is an example

# Problems with Weak Perfect Bayesian Equilibrium

## Example



# Problems with Weak Perfect Bayesian Equilibrium

## Example

- Claim: The following is a WPBE of this game
  - $\mu_1(S_1) = \mu_1(S_2) = 0.5$
  - $\mu_2(S_1|d) = 0.1$, $\mu_2(S_2|d) = 0.9$
  - $\sigma_1(u) = 1$
  - $\sigma_2(l) = 1$
- Why?
  - Player 2 optimizing given beliefs

$$u(l|\mu_2) = 5,\ u(r|\mu_2) = 0.9.2 + 0.1.10 = 2.8$$

  - Player 1 optimizing given beliefs and $\sigma_2$

$$\begin{aligned} u(u|\mu_1, \sigma_2) &= 2 \\ u(d|\mu_1, \sigma_2) &= 0 \end{aligned}$$

# Problems with Weak Perfect Bayesian Equilibrium

## Example

- Beliefs are generated by Bayes rule wherever possible

$$\mu_1(S_1) = \mu_1(S_2) = 0.5$$

- **But**, notice that P2's information set is never reached, so we can use Bayes' rule

$$\mu_2(S_1|d) = \frac{\mu_2(S_1 \cap d)}{\mu_2(d)}$$

  $\mu_2(d) = 0!$

- Because we can't use Bayes' rule, WPB does not constrain beliefs!
  - Anything goes
- But these beliefs don't seem sensible

# Sequential Equilibrium

- There are lots of refinement concepts designed to deal with the problem of 'unreasonable' beliefs off the equilibrium path
- The first one we are going to define employes a trick similar to trembling hand perfect NE
- This approach is particularly well suited to dealing with this problem:
  - **Problem:** Some information sets are reached with zero probability and so we can't use Bayes rule to pin down beliefs
  - **Solution:** Use completely mixed strategies to ensure that every information set is reached with positive probability
- This is the notion of **sequential equilibrium**

# Sequential Equilibrium

**Definition**

Strategies $\sigma$ and beliefs $\mu$ form a sequential equilibrium if

1. $\sigma$ is sequentially rational given $\mu$
2. There is a sequence of completely mixed strategies $\sigma^k \to \sigma$ such that the resulting beliefs $\mu^k \to \mu$, where $\mu^k$ are the beliefs derived from $\sigma^k$ using Bayes rule

# Sequential Equilibrium

- How does this solve the problem we had in the previous example?
- Note that P1 cannot condition their strategy on the state $S$
- Thus $\sigma_1(u|S1) = \sigma_1(u|S2)$
- For any completely mixed strategy we therefore have

$$\begin{aligned}
\mu_2(S_1|d) &= \frac{\mu_2(S_1 \cap d)}{\mu_2(d)} \\
&= \frac{0.5.\sigma_1(u|S1)}{0.5.\sigma_1(u|S1) + 0.5.\sigma_1(u|S2)} \\
&= \frac{1}{2}
\end{aligned}$$

# Sequential Equilibrium

- So, for any sequential equilibrium
  - $\mu_2(S_1|d) = 0.5$
  - Implies that

$$u(l|\mu_2) = 5,\ u(r|\mu_2) = 0.5.2 + 0.5.10 = 6$$

    and so $\sigma_2(r) = 1$
  - And so

$$\begin{aligned} u(u|\mu_1, \sigma_2) &= 2 \\ u(d|\mu_1, \sigma_2) &= 5 \end{aligned}$$

  - $\sigma_1(d) = 1$