G5212: Game Theory

Mark Dean

Spring 2017

Why Game Theory?

- So far your microeconomic course has given you many tools for analyzing economic decision making
- What has it missed out?
- Sometimes, economic agents **interact directly**
    - Two people bidding for the same item on ebay
    - Two people working together on a joint project
    - Two generals deciding where to position their armies
    - Two firms setting prices for similar products
- Key feature: the outcome for each person depends on their actions *and the actions of the other person*

Why Game Theory?

- In such cases, optimization (on its own) will not get us very far
  - Best bid of auctioneer 1 depends on bid of auctioneer 2
  - Best bid of auctioneer 2 depends on bid of auctioneer 1
- Need some way of solving both problems together
- This (basically) is what game theory studies
- One of the big theoretical and practical success stories of microeconomics
- Applied to mating displays of birds, banking crises, spectrum auctions, kidney exchanges, insurance, school choice, political platforms, sport, war, kin selection, etc, etc etc

The Plan (For The Course)

- Part 1: Game Theory (until March 8th)
- Focus on tools
    - Static games of complete information
    - Dynamic games of complete information
    - Games of incomplete information
    - Solution concepts and refinements
- With a few applications
    - Bargaining
    - Auctions
    - Experimental evidence

The Plan (For The Course)

- Part 2: Information Economics
- Focus on applications in the face of *asymmetric information*
- Example 1: Signalling
  - PhD programs want to recruit people of high ability
  - But they cannot observe ability directly
  - Can education be used by high ability candidates to signal that they have high ability?
- Example 2: Moral Hazard
  - A boss wants to encourage their worker to work hard
  - But they cannot observe effort directly, only outcomes (which have a random component)
  - How should they design their incentive scheme?

The Plan (for today)

- A gentle introduction!
- Talk through some ‘classic’ games
- Formal definition of a game
- Mixed strategies

Matching Pennies

Example

Matching Pennies

		Bob H	Bob T
Anne	H	$+1, -1$	$-1, +1$
	T	$-1, +1$	$+1, -1$

- Two players (Ann and Bob) each reveal a penny showing heads or tales
- If the pennies match then Bob pays Ann a dollar
- If not, Ann pays Bob a dollar
- This is the matrix form of this game
- Other applications?

Prisoner's Dilemma

Example

Prisoner's Dilemma

		Bob	
		Confess	Don't Confess
Anne	Confess	$-6, -6$	$0, -9$
	Don't Confess	$-9, 0$	$-1, -1$

- Probably the most famous game in all of game theory
- Classic story of two prisoners who must decide whether or not to confess
- But many other (more economically interesting) applications

Example

One application is the partnership game: effort E produces an output of 6 at a cost of 4, with output shared equally; shirking S produces an output of 0 at a cost of 0.

		Bob: S	Bob: E
Anne	S	$0, 0$	$3, -1$
	E	$-1, 3$	$2, 2$

The canonical form of the prisoner's dilemma is given by

		Bob: Confess	Bob: Don't
Anne	Confess	P, P	T, S
	Don't	S, T	R, R

where T(emptation)>R(eward)>P(unishment)>S(ucker)

Defining a Game

- A normal form game consists of 3 elements

1. The players
2. The actions that each player can take
3. The payoffs associated with each set of actions

- Notice that we are initially making some 'hidden' assumptions
    - Players move at the same time
    - All payoffs are known to all players
- Later in the course we will relax these assumptions, and so a description of the game will also include
    - The sequence of play
    - Who knows what

Definition

An n-player normal (or strategic) form game G is an n-tuple $\{(S_1, u_1), ..., (S_n, u_n)\}$, where for each i
(1) S_i is a nonempty set, called i's strategy space, and
(2) $u_i : \prod_{k=1}^{n} S_k \to \mathbb{R}$ is called i's payoff function.

- **Notation**
  - $S := \prod_{k=1}^{n} S_k$
  - $s := (s_1, ..., s_n) \in S$
  - $S_{-i} := \prod_{k \neq i} S_k$
  - $(s'_i, s_{-i}) := (s_1, ..., s_{i-1}, s'_i, s_{i+1}, ..., s_n)$

Definition

A normal form game is simply a vector-valued function $u : S \to \mathbb{R}^n$

Example

Second-Price Sealed-Bid Auction. A seller has one indivisible object. There are n bidders with respective valuations $0 \leq v_1 \leq \cdots \leq v_n$ for the object (which are common knowledge). The bidders simultaneously submit bids. The highest bidder wins the object and pays the *second highest* bid. In the case of a tie all winning bidders are equally likely to have their bid accepted.

- Players: 1,....,n
- Strategies: $S_i \in [0, \infty)$
- Payoffs: Given a profile of bids, s, let $W(s) \equiv \{k : \forall j, s_k \geq s_j\}$ be the set of highest bidders. Then the game is simply the following:

$$u_i(s_i, s_{-i}) = \begin{cases} v_i - \max_{j \neq i} s_j & \text{if } s_i > \max_{j \neq i} s_j \\ \frac{1}{|W(s)|}(v_i - s_i) & \text{if } s_i = \max_{j \neq i} s_j \\ 0 & \text{if } s_i < \max_{j \neq i} s_j. \end{cases}$$

Example

Cournot Duopoly. There are two firms, call them 1 and 2, producing perfectly substitutable products: market demand is $P(Q) = \max\{a - Q, 0\}$, $Q = q_1 + q_2$. The cost of producing q_i is given by $C(q_i) = cq_i$, $0 < c < a$. The two firms choose quantities simultaneously.

- Players: 1,2
- Strategies $S_i \in [0, \infty)$.
- Payoffs

$$u_i(q_1, q_2) = (P(q_1 + q_2) - c)\, q_i.$$

Example

There are three players $i = 1, 2, 3$ and two candidates a and b which they can vote for. The voting rule is the majority rule. Voters' preferences are as follows

1	2	3
a	b	b
b	a	a

A player receives a payoff of 1 if his favorite candidate wins and a payoff of 0 if his less favorite candidate wins.

- Players: 1,2,3
- Strategies $S_i = \{a, b\}$
- Payoffs (for Player 1):

$$\begin{array}{l} u_1(a,a,a) = 1 \\ u_1(a,a,b) = 1 \\ u_1(a,b,a) = 1 \\ u_1(a,b,b) = 0 \end{array}, \quad \begin{array}{l} u_1(b,a,a) = 1 \\ u_1(b,a,b) = 0 \\ u_1(b,b,a) = 0 \\ u_1(b,b,b) = 0 \end{array}$$

Mixed Strategies

- Consider again the matching pennies game
- Here is another action Bob could take: rather than put the coin down H or T, he could flip it, and play whichever way the coin falls
- This is a new strategy: it is not H or T, but a 50% chance of H and a 50% chance of T
- More generally, we might like to extend the player's strategy space to allow them to randomize between pure strategies
- These are 'Mixed Strategies'
- They will be useful going forward....

Mixed Strategies

Definition

Suppose $\{(S_1, u_1), ..., (S_n, u_n)\}$ is an n-player normal-form game. A **mixed strategy** for player i is a probability distribution over elements of S_i, denoted by $\sigma_i \in \Delta(S_i)$. Strategies in S_i are called **pure strategies**.

Remarks

- In most cases, we assume S_i is finite. Then $\sigma_i : S_i \rightarrow [0, 1]$ s.t. $\sum_{s_i \in S_i} \sigma_i(s_i) = 1$.
- Where S_i is not countable there are some technical concerns about defining mixed strategies
  - Need to define an appropriate σ-algebra, etc
  - We will not worry about this

Mixed Strategies

Extend u_i to $\prod_{j=1}^{n} \Delta(S_j)$ by taking expected values. If S_i is finite:

$$u_i(\sigma_1, \ldots, \sigma_n) := \sum_{s_1 \in S_1} \ldots \sum_{s_n \in S_n} u_i(s_1, \ldots, s_n)\, \sigma_1(s_1)\, \sigma_2(s_2) \cdots \sigma_n(s_n).$$

Notation:

$$\begin{aligned} u_i(s_i, \sigma_{-i}) &:= \sum_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}) \prod_{j \neq i} \sigma_j(s_j) \\ u_i(\sigma_i, \sigma_{-i}) &:= \sum_{s_i \in S_i} u_i(s_i, \sigma_{-i})\, \sigma_i(s_i) \end{aligned}$$

- Note that we are implicitly assuming risk neutrality, or assuming that payoffs are in utility units

Mixed Strategies

- Note that a game

$$\{(S_1, u_1), ..., (S_n, u_n)\}$$

  in which we don't allow mixed strategies induces another game

$$\{(\Delta(S_1), u_1), ..., (\Delta(S_n), u_n)\}$$

  when mixed strategies are allowed
- Do mixed strategies mean that all distributions over strategies are allowed?
- No, because we don't allow for correlation
  - $\prod_{j=1}^{n} \Delta(S_j) \neq \Delta\left(\prod_{j=1}^{n} S_j\right) = \Delta(S)$
  - $(\sigma_1, ..., \sigma_{i-1}, \sigma_{i+1}, ..., \sigma_n) =: \sigma_{-i} \in \prod_{j \neq i} \Delta(S_j) \neq \Delta(S_{-i})$
- Example?

Summary

- Key things from today
  - Understand what a game is
  - Understand how to translate a story into a game
  - Understand what mixed strategies are
  - Understand how to translate a game in pure strategies into a game with mixed strategies