G5212: Game Theory

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The Story So Far...

- Last week we
  - Introduced the concept of a dynamic (or extensive form) game
  - The strategic (or normal) form of that game
- In terms of solution concepts we
  - Described the Nash equilibrium of a dynamic game as the Nash equilibrium of the associated normal form game
  - Showed that some NE were 'non-credible'
  - Introduced 'backward induction' as a way of identifying credible NE
  - Showed that this was the same as assuming 'Common Knowledge of Sequential Rationality'

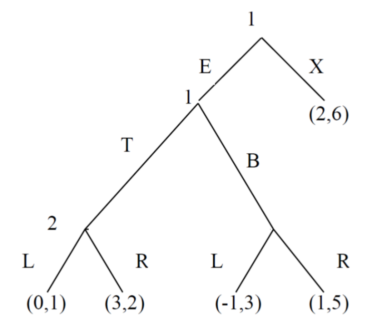
This Lecture

- This lecture we will
  - Extend the concept of backward induction to that of subgame perfect Nash equilibrium
  - Discuss a potential problem with backward induction
  - Apply SPNE to bargaining games

Limits on Backward Induction

Example

A warm up

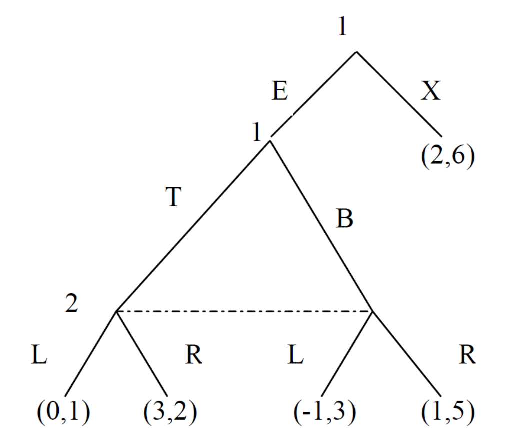


- What are the Nash Equilibria of this game?
- What survives Backward Induction?

Limits on Backward Induction

Example

How to do backward induction for this game?



Limits on Backward Induction

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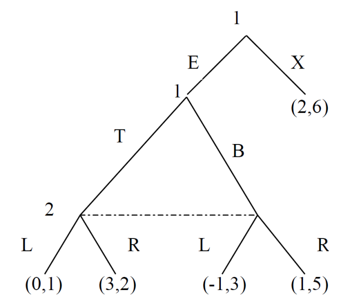
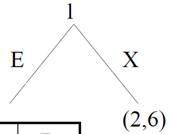




	L	R
T	0,1	3,2
B	-1,3	1,5

Subgame Perfect (Nash) Equilibrium

- There are two cases in which backwards induction cannot be applied

1. If the game has an infinite horizon
2. If it is a game of incomplete information

- To tackle such cases, we need a sightly more sophisticated concept
  - Subgame Perfect Nash Equilibrium

Defining A Subgame

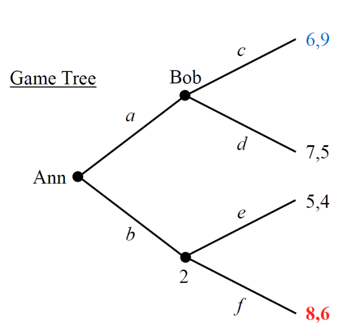
Definition

A subgame is any part (a subset) of a game that meets the following criteria

1. It has a single initial node that is the only member of that node's information set (i.e. the initial node is in a singleton information set).
2. If a node is contained in the subgame then so are all of its successors.
3. If a node in a particular information set is in the subgame then all members of that information set belong to the subgame.

Defining A Subgame

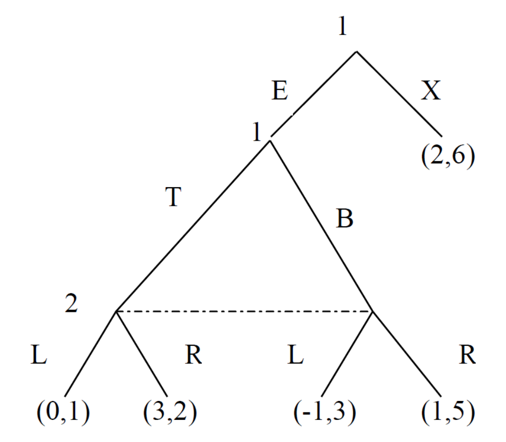
Example



- How many subgames does this game have?

Defining A Subgame

Example



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Defining A Subgame

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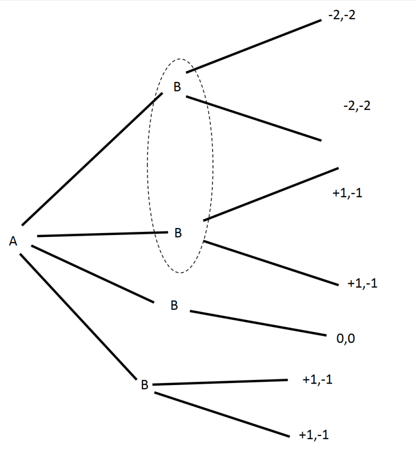


- How many subgames does this game have?

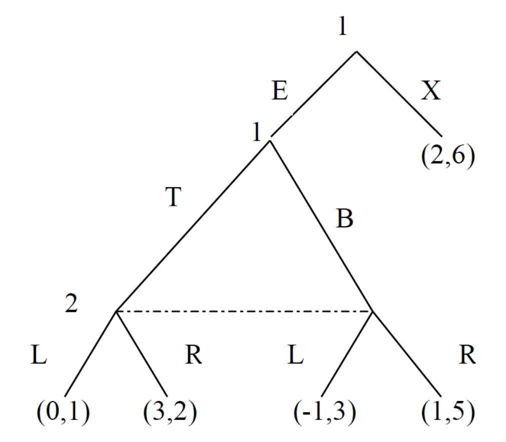
Subgame Perfect (Nash) Equilibrium

- **Subgame Perfect (Nash) Equilibrium** (SPNE) is a refinement of Nash equilibrium
- A strategy profile forms a SPNE if:
  - It is a Nash Equilibrium
  - When restricted to any subgame, it forms a Nash equilibrium for that subgame.
- In finite games of complete information, set of SPNE is the set of strategy profiles one gets from backward induction
- But the concept of SPNE can also be applied to infinite games and games of incomplete information

Subgame Perfect Nash Equilibrium - Example

Example

SPNE is a NE in each game



Subgame Perfect Nash Equilibrium - Example

- Subgame 1: The whole game:

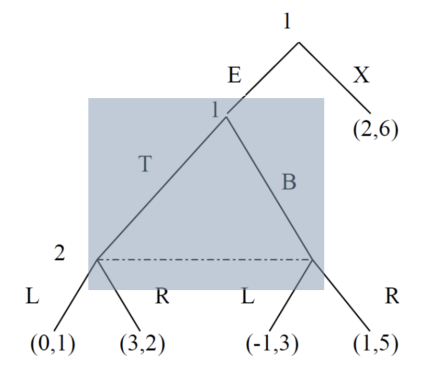
	L	R
XT	$(2, 6)$	$(2, 6)$
XB	$(2, 6)$	$(2, 6)$
LT	$(0, 1)$	$(3, 2)$
LB	$(-1, 3)$	(1.5)

- Three NE: (XT, L), (XB, L) and (LT, R)

Subgame Perfect Nash Equilibrium - Example

Example

SPNE is a NE in each game



- Subgame 2

Subgame Perfect Nash Equilibrium - Example

- Subgame 2:

	L	R
T	$(0, 1)$	$(3, 2)$
B	$(-1, 3)$	(1.5)

- One NE: (L, R)

Subgame Perfect Nash Equilibrium - Example

- Thus (LT, R) is the only NE in the first game that also induces a NE in all other subgames
  - Kills (XT, L) and (XB, L)
- Allows us to carry over the backward induction reasoning into settings where backward induction cannot be applied

SPNE and One Shot Deviation Principle

- It seems like there is a **lot** to check when it comes to determining whether a strategy is a SPNE
- Luckily, we can use a handy trick
- The **one shot deviation principle**

Definition

For any strategy in an extensive form game, a one-shot deviation is a strategy that varies only in the action taken at the initial node

Theorem

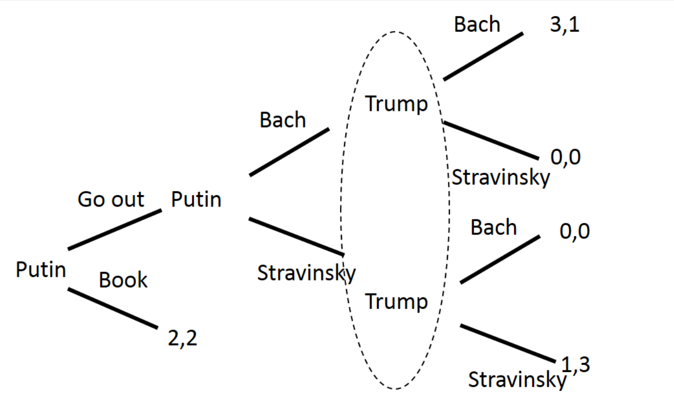
For any finite game, a strategy profile $(s_1, ..., s_n)$ *is a SPNE if and only if for every player and every subgame there is no one shot deviation that leads to a higher payoff*

- This will be particularly handy when we talk about repeated games in the next lecture

A Potential Problem with SPNE

Example

BoS with outside option



A Potential Problem with SPNE

- What are the SPNE of this game?
- Subgame 1: The whole game:

	B	S
GB	$(\mathbf{3},\mathbf{1})$	$(0,0)$
GS	$(0,0)$	$(1,3)$
BB	$(2,2)$	$(\mathbf{2},\mathbf{2})$
BS	$(2,2)$	$(\mathbf{2}.\mathbf{2})$

- Two equilibria: (GB,B), (BB,S) and (BS,S)

A Potential Problem with SPNE

- What are the SPNE of this game?
- Subgame 2: The BoS game

	B	S
GB	$(\mathbf{3}, \mathbf{1})$	$(0, 0)$
GS	$(0, 0)$	$(\mathbf{1}, \mathbf{3})$

- Two equilibria (B, B) and (S, S)

A Potential Problem with SPNE

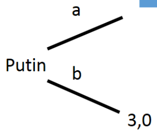
- Two SPNE
  - (GB, B) and (BS, S)
- Are both equally convincing?
- Arguably not
  - Imagine that Trump finds himself playing the BoS game
  - Is it reasonable to think that Putin has played S?
  - Probably not. Putin could have guaranteed himself 2 by playing B
  - Why would he enter a subgame and play in a manner in which he is only going to get 1
  - Arguably (BS, S) is not reasonable, despite being SPNE
- This is an example of **forward induction** reasoning

Backward Induction vs Forward Induction

Example

Trump

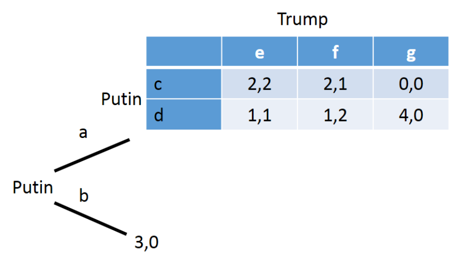
Putin	e	f	g
c	2,2	2,1	0,0
d	1,1	1,2	4,0



- What are the SPNE of this game?
    - (c, e) is the only NE of the second game
    - (bc, e) is the unique SPNE

Backward Induction vs Forward Induction

Example



- What about 'forward induction'?
    - If Trump finds himself at the second stage game, what should he assume?
    - Knows that Putin can guarantee himself 3
    - Must think he is getting 4 - and so playing d
    - Best response is f

Backward Induction vs Forward Induction

- Forward Induction is not a refinement of SPNE
  - Central to the Forward Induction concept is that previous play tells you something about future play
  - Subgames cannot be treated in isolation
- Despite intuitive plausibility, formalizing notion of Forward Induction has proved tricky
  - Beyond the scope of this course
  - For those interested see: Govindan, Srihari, and Robert Wilson. "On forward induction." Econometrica 77.1 (2009): 1-28.