G5212: Game Theory

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  - Players can be uncertain about what the other player will do
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  - Payoffs, strategies and players all common knowledge

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    - Players can be uncertain about what the other player will do
    - But they **cannot** be uncertain about the game that they are playing
    - Payoffs, strategies and players all common knowledge
- This is very restrictive
    - Many situations that we can't analyze
    - Particularly situations of asymmetric information
    - This is essentially what the second half of the course will be about

Motivation

- Example 1: Signalling
  - PhD programs want to recruit people of high ability
  - But they cannot observe ability directly
  - Can education be used by high ability candidates to signal that they have high ability?

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- Example 1: Signalling
    - PhD programs want to recruit people of high ability
    - But they cannot observe ability directly
    - Can education be used by high ability candidates to signal that they have high ability?
    - Employer uncertain about the quality of the candidate
        - How good the candidate will be if they hire them (own payoff)
        - Cost of education to the candidate (other's payoff)
    - Information is asymmetric
        - Candidates know their ability, firms do not

Motivation

- Example 2: Auctions
  - An item is being auctioned on ebay
  - Highest bidder wins the auction and gets the item
  - Each bidder has a different valuation for the item
  - How much should each person bid?

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- Example 2: Auctions
  - An item is being auctioned on ebay
  - Highest bidder wins the auction and gets the item
  - Each bidder has a different valuation for the item
  - How much should each person bid?
  - Each bidder is uncertain about the value of the item to other bidders (other's payoff)
  - (Possibly) the number of bidders
  - Information is asymmetric
    - Each bidder knows their own valuation, not the valuation of everyone else

Motivation

- Example 3: Currency Attacks
  - The strength of a currency is uncertain
  - There are a number of speculators deciding whether to attack the currency
  - The number who need to attack to bring the currency down depends on its strength
  - Each player receives a signal about the strength of the currency

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- Example 3: Currency Attacks
  - The strength of a currency is uncertain
  - There are a number of speculators deciding whether to attack the currency
  - The number who need to attack to bring the currency down depends on its strength
  - Each player receives a signal about the strength of the currency
  - Each bidder uncertain about the strength of the currency (own and others payoff)
  - Also uncertain about what signal others have got (information of others)
  - Information is asymmetric
    - Each bidder knows their own signal

A Worked Example

- Cournot Competition with unknown types

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- Two firms $i = 1, 2$. Each firm i's cost $c_i \in \{c_i^H, c_i^L\}$ is i's private info.

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- Probability of each type of firm

$$p(c_1^H, c_2^H) = p(c_1^L, c_2^L) = \frac{1}{2}\alpha$$

$$p(c_1^H, c_2^L) = p(c_1^L, c_2^H) = \frac{1}{2}(1-\alpha)$$

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- Each firm chooses an amount to produce **dependent on type**

$$q_i^H \in [0, \infty),\ q_i^L \in [0, \infty)$$

- Payoffs for cost $c_i \in \left\{c_i^H, c_i^L\right\}$ and $q_i \in \left\{q_i^H, q_i^L\right\}$

$$q_i \left(P(Q) - c_i\right)$$

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- Let's think about the behavior of player 1
- Fix a strategy for player 2: $\{q_2^H, q_2^L\}$
- If player 1 has costs c^H what is the probability that player 2 is H?
- Use Bayes' rule

$$
\begin{aligned}
P(c_2^H|c_1^H) &= \frac{P(c_2^H \cap c_1^H)}{P(c_1^H)} \\
&= \frac{\frac{1}{2}\alpha}{\frac{1}{2}\alpha+\frac{1}{2}(1-\alpha)} = \alpha
\end{aligned}
$$

A Worked Example

- If player 1 is of type c^H their optimization problem is therefore

$$\max_q \left[\alpha(1-(q_2^H+q)-c_1^H)q+(1-\alpha)(1-(q_2^L+q)-c_1^H)q\right].$$

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- FOC:

$$\alpha\left(1-c_1^H-q_2^H-2q\right)+(1-\alpha)\left(1-c_1^H-q_2^L-2q\right)=0$$

or

$$q=\frac{1-c_1^H-\alpha q_2^H-(1-\alpha)q_2^L}{2}$$

Cournot competition with private cost

- We can repeat the exercise for the case in which player 1 has c^L

$$P(c_2^H|c_1^L) = \frac{P(c_2^H \cap c_1^L)}{P(c_1^L)} = \frac{\frac{1}{2}(1-\alpha)}{\frac{1}{2}\alpha+\frac{1}{2}(1-\alpha)} = (1-\alpha)$$

- So they must maximize

$$\max_q \left[(1-\alpha)(1-(q_2^H+q)-c_1^L)q+\alpha(1-(q_2^L+q)-c_1^L)q\right].$$

- Giving first order conditions

$$q = \frac{1-c_1^L-(1-\alpha)q_2^H-\alpha q_2^L}{2}$$

Cournot competition with private cost

- Repeating the exercise for Player 2 gives 4 equations and 4 unknowns

$$q_1^H = \frac{1 - c_1^H - \alpha q_2^H - (1-\alpha)q_2^L}{2}$$

$$q_1^L = \frac{1 - c_1^L - (1-\alpha)q_2^H - \alpha q_2^L}{2}$$

$$q_2^H = \frac{1 - c_2^H - \alpha q_1^H - (1-\alpha)q_1^L}{2}$$

$$q_2^L = \frac{1 - c_2^L - (1-\alpha)q_1^H - \alpha q_1^L}{2}$$

- Which can be solved to get the solution to this problem

Bayesian Games

- Let's now try to formalize what we just did
- First the set up
    - N: Set of players (firms)
    - $T = \{T_n\}_{n \in N}$: Types space for each player (possible cost of each firm)
    - $P \in \Delta(T)$: prior probabilities over types (α)
    - $S = \{S_n\}_{n \in N}$: Strategy space for each player (output levels)
    - $u : S \times T \to \mathbb{R}^N$: Payoffs function for each player (profits)

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- Notes
  - P the same for each player - no agreeing to disagree
  - S_n doesn't depend on t_n
  - If we restrict u_n to only depend on t_n this is the case of **independent values**

Bayesian Games

- Solution
- Every play picked a (mixed) strategy $\sigma_n^* : T_n \to \Delta(S_i)$ (production levels **conditional on type)**
- In order to maximize

$$\mathbb{E}\left[u_n\left(\sigma_n, \sigma_{-n}^*\left(t_{-n}\right), t\right) | t_n\right]$$

- Where

$$\begin{aligned} & \mathbb{E}\left[u_n\left(\sigma_n, \sigma_{-n}^*\left(t_{-n}\right), t\right) | t_n\right] \\ = & \sum\nolimits_{t_{-n} \in T_{-n}} u_n\left(\sigma_n, \sigma_{-n}^*\left(t_{-n}\right), t\right) p\left(t_{-n} | t_n\right). \end{aligned}$$

$$p\left(t_{-n} | t_n\right) = \frac{p\left(t_{-n}, t_n\right)}{p(t_n)}$$

- This is the **Bayesian Nash Equilibrium** of the game

Bayesian Games

Example

Consider a modified BOS game in which the type of the S player is unknown
With prob 1/2 the payoff matrix is 1/2 (friendly)

		S: Bach	S: Stravinsky
B	Bach	2, 1	0, 0
	Stravinsky	0, 0	1, 2

With prob 1/2 the payoff matrix is (unfriendly)

		S: Bach	S: Stravinsky
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	Stravinsky	0, 1	1, 0

Bayesian Games

- What is the best response of the S player?

Bayesian Games

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  - $BR(B|Friendly) = B$
  - $BR(S|Friendly) = S$
  - $BR(B|Unfriendly) = S$
  - $BR(S|Unfriendly) = B$

Bayesian Games

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Bayesian Games

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- Remember, the B player best responds to a **strategy** of the S player
    - Action conditional on type
    - e.g. BS : Bach if friendly, Stravinsky if unfriendly

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- Remember, the B player best responds to a **strategy** of the S player
  - Action conditional on type
  - e.g. BS : Bach if friendly, Stravinsky if unfriendly
- Payoff for playing B against BS is $\frac{1}{2}2 + \frac{1}{2}0$
- So
  - $BR(BB) = B$
  - $BR(BS) = B$
  - $BR(SB) = B$
  - $BR(SS) = S$

Bayesian Games

- Can it be part of a NE for Stravinsky to play S?

Bayesian Games

- Can it be part of a NE for Stravinsky to play S?
  - No
  - Best response to S is SB
  - Best response to SB is B

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  - No
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