G5212: Game Theory

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Adverse Selection

- We are now going to go back to the Adverse Selection framework
  - Mechanism Design with 1 agent
  - Though that agent may be of many types
- Note that implementation in dominant strategies is now the only game in town
  - As there is only one player, NE and BNE are both the same as dominant strategies

The Taxation Principle

- In this setting, there is a second handy general result to go with revelation principle
    - Recall RP says that any $y(\theta)$ that can be implemented can be implemented using a direct truth telling mechanism
- Assume that preferences are quasi-linear

$$u_i(y, t|\theta) = u_i(y|\theta) - t$$

- Then any $y(\theta)$ that can be implemented can be implemented using a *nonlinear tariff*
    - The agent is offered a menu of possible outcomes Y
    - For picking $y \in Y$ they get charged some $t(y)$

The Taxation Principle

- Proof:
  - Take any implementable y
  - We know it can be implemented by a truth telling direct mechanism
  - Note that, if $y(\theta) = y(\theta')$ then it must be the case that $t(\theta) = t(\theta')$
    - If (say) $t(\theta) < t(\theta')$ then θ' would always report θ
  - Thus, the principal can set

$$Y = \cup_{\theta \in \Theta} y(\theta)$$

    with $t(y) = t(\theta) | y = y(\theta)$ which is unique
  - For type θ

$$\begin{aligned} u(y(\theta)|\theta) - t(y(\theta)) &= u(y(\theta)|\theta) - t(\theta) \geq \\ u(y(\theta'), \theta) - t(\theta') &= u(y(\theta'), \theta) - t(y(\theta')) \end{aligned}$$

Price Discrimination

- The first Adverse Selection problem that we are going to attack in any detail is price discrimination
  - Different types of New Yorkers have different preferences over Mets tickets
  - Some want to go to the game for a cheap day out (Cheapskates)
  - Some love baseball, and really want the best seats (Afficionados)
  - The Mets do not know which person is which, so can't offer different prices to Cheapskates and Afficianados
  - How do they design their prices so that Afficionados buy the good seats and the Cheapskates get the nosebleeds?
  - Does this maximize profit?

The Agent

- The Salanie book does this with wine
- I would like to do it with Tea instead, but because I am sad about Brexit we will stick to wine
    - Agents come of two types (we will relax this later)
    - θ_1 (frugal) and θ_2 (sophisticated): $\theta_1 < \theta_2$
    - Wine varies in quality q
    - Utility is given by

$$U(q, t|\theta) = u(q|\theta) - t = \theta q - t$$

    - Probability of $\theta_1 = \pi_1$
- We will assume that every type always has the option to buy no wine, and get utility zero

The Agent

- Notice that this utility function implies that
  - Utility is always increasing in quality and decreasing in price regardless of quality
  - But, for $\theta' > \theta$

    $$u(q|\theta') - u(q|\theta) = (\theta' - \theta)q$$

    increases in q
- Sophisticated consumer is always prepared to pay more than the frugal one for an increase in quality

The Principal

- Monopolist with cost function $C(q)$
  - Twice differentiable
  - Strictly convex
- Utility (profit) given by

$$t - C(q)$$

The First Best Solution

- Let's first solve this problem assuming the monopolist can observe different types
- We will often use this as a benchmark
- Called 'First best solution'
- The principal will propose different quality/tariff pairs for the two types
- For type i, the principal will maximize

$$t_i - C(q_i)$$

subject to

$$\theta_i q_i - t_i \geq 0$$

The First Best Solution

- First, note that the principal will always set $\theta_i q_i = t_i$
- So we can rewrite the problem as

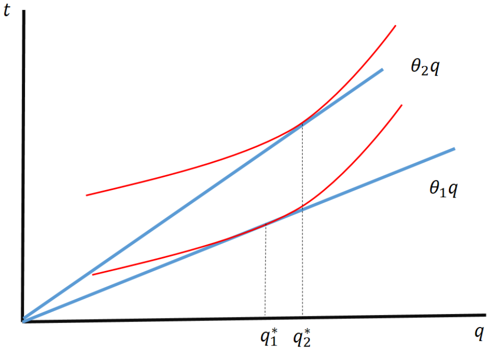
$$\max_q \theta_i q - C(q_i)$$

- Will pick q_i^* such that

$$C'(q_i^*) = \theta_i$$

  and $t_i^* = \theta_i q_i^*$
- Choose q_i to maximize the surplus of θ_i, then extract all the surplus using t_i

The First Best Solution



- Red lines are iso-profit lines
- Note that more profit is made from type 2 (bigger surplus)

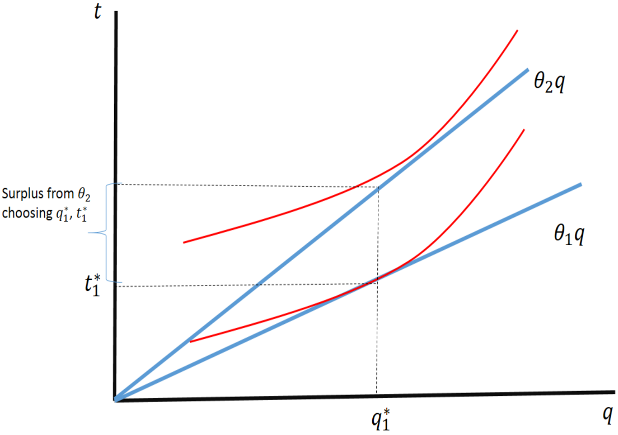
Asymmetric Information

- So what if the principal cannot observe different types
  - Only knows π_1
  - Equivalently not allowed to condition prices on types?
- Will the first best solution work?
  - i.e. if we offer contracts $\{q_1^*, t_1^*\}$ and $\{q_2^*, t_2^*\}$ will the right types of agent buy the right quantity?
- No!

$$\begin{aligned} \theta_2 q_1^* - t_1^* &= \theta_2 q_1^* - \theta_1 q_1^* \\ &= (\theta_2 - \theta_1)\, q_1^* \\ &> 0 = \theta_2 q_2^* - t_2^* \end{aligned}$$

- Type 2 will pretend to be type 1

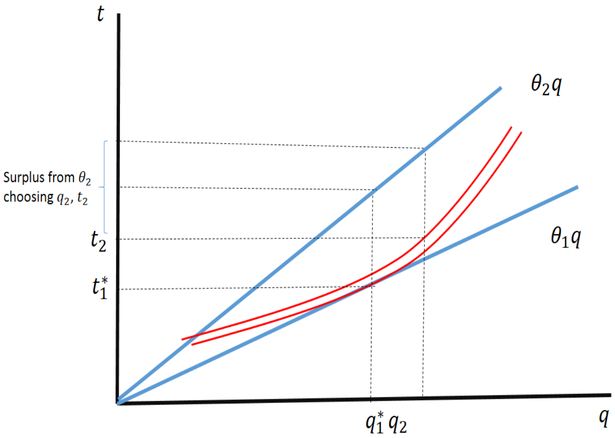
Asymmetric Information



Asymmetric Information

- Can we do better than this?
- Yes

Asymmetric Information



Asymmetric Information

- The principal will do better offering $\{q_2, t_2\}$ with $\{q_1^*, t_1^*\}$ than they would offering $\{q_2^*, t_2^*\}$
    - Under this pair, θ_1 will choose $\{q_1^*, t_1^*\}$ and θ_2 will choose $\{q_2, t_2\}$
    - Profit from $\{q_2, t_2\}$ higher than from $\{q_1^*, t_1^*\}$

The Constrained Optimization Problem

- A more general statement of the problem:

$$\max_{q_1,t_1,q_2,t_2} \pi_1[t_1 - C(q_1)] + \pi_2[t_2 - C(q_2)]$$

- Subject to

$$\begin{aligned} \theta_1 q_1 - t_1 &\geq \theta_1 q_2 - t_2 \\ \theta_2 q_2 - t_2 &\geq \theta_2 q_1 - t_1 \\ \theta_1 q_1 - t_1 &\geq 0 \\ \theta_2 q_2 - t_2 &\geq 0 \end{aligned}$$

Asymmetric Information

- Individual rationality constraints: Has to be better for each party to choose their contract rather than choose no contract

$$\begin{aligned} \theta_1 q_1 - t_1 &\geq 0 \\ \theta_2 q_2 - t_2 &\geq 0 \end{aligned}$$

- Incentive compatibility constraints: Has to be better to choose your own contract

$$\begin{aligned} \theta_1 q_1 - t_1 &\geq \theta_1 q_2 - t_2 \\ \theta_2 q_2 - t_2 &\geq \theta_2 q_1 - t_1 \end{aligned}$$

Asymmetric Information

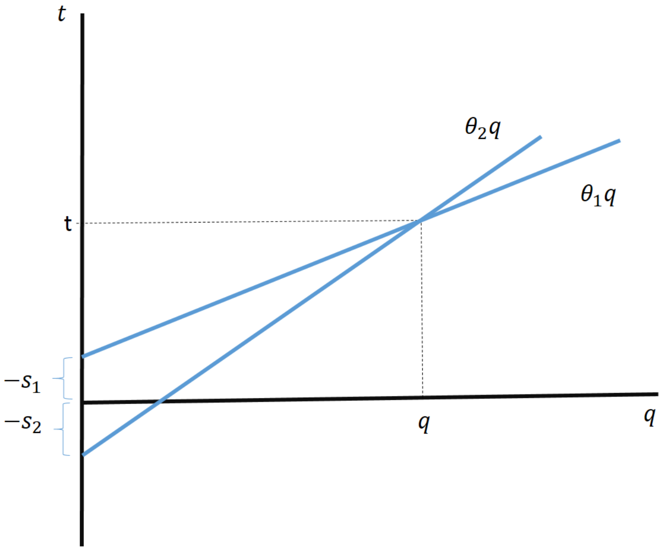
- What do these contraints look like?
- Let s_i be the surplus that person i gets from a particular contract

$$s_i = \theta_i q - t$$

- This gives 'iso-surplus' lines

$$t = \theta_i q - s_i$$

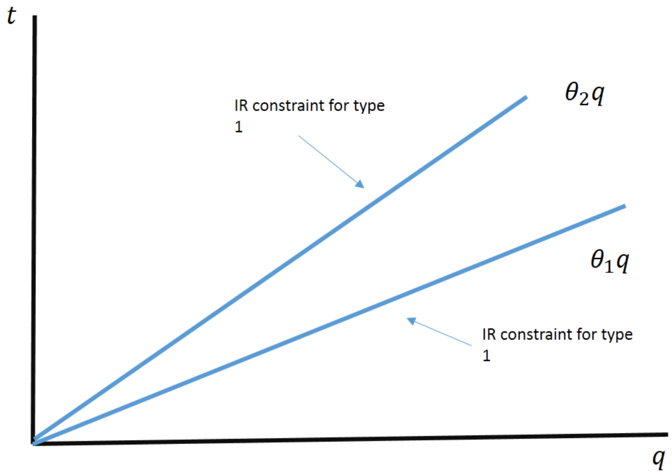
Asymmetric Information



Asymmetric Information

- What do these constrainst look like?
- First, the IR constraints

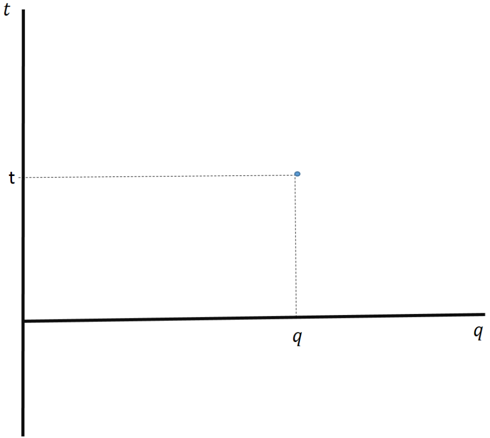
IR Constraints



Asymmetric Information

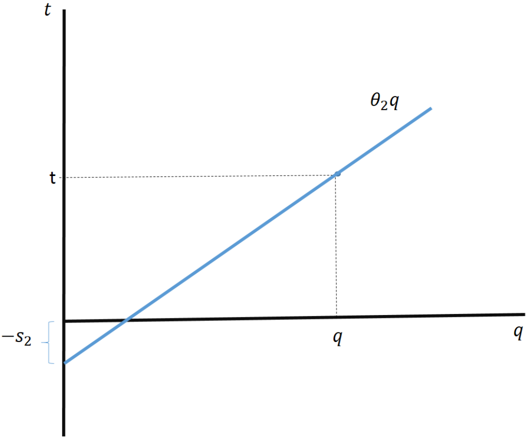
- What do these constrainst look like?
- First, the IR constraints
- Now the IC constrainst

IC Constraints



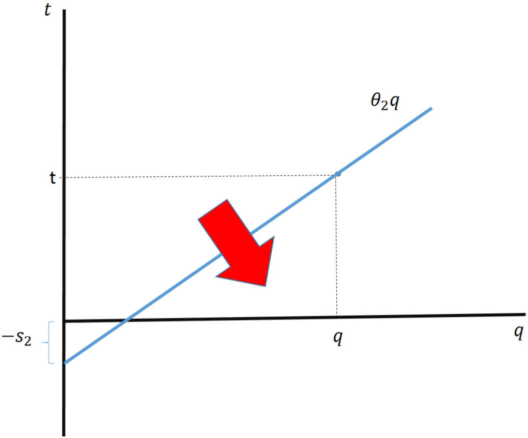
- Say this is the contract offered to type 1

IC Constraints



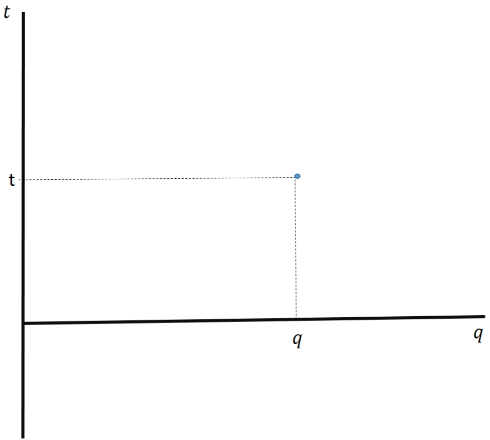
- This is the surplus for type 2 from choosing this contract

IC Constraints



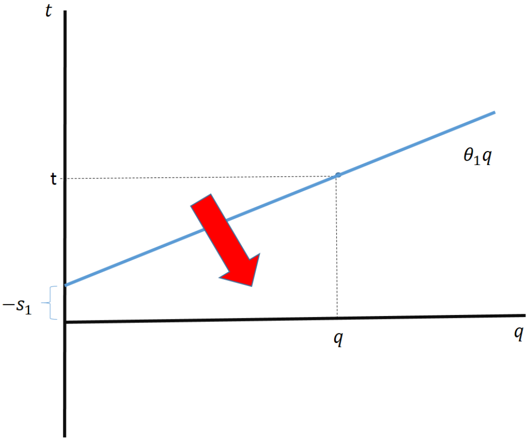
- So the contract offered to type two must offer more surplus

IC Constraints



- Similarly, if this were the contract offered to type 2

IC Constraints

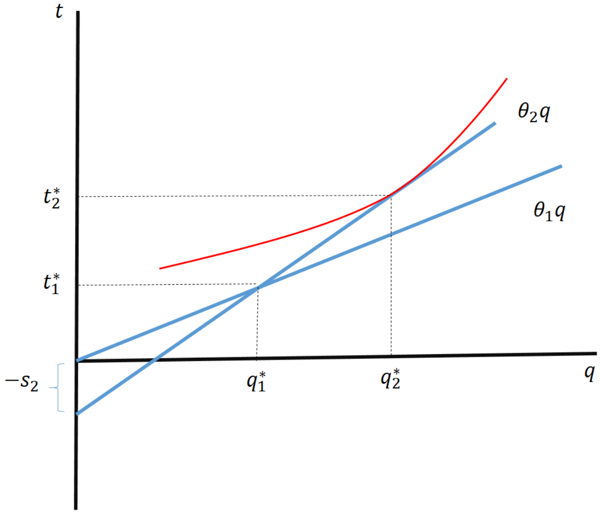


- Then the contract offered to type 1 must give them more surplus

Which Constraints Bind?

- Do all of these constrains bind at the same time?
- No!
- We will show that at the optimal contract
  - IR1 will bind
  - IC2 will bind
  - IR2 and IC1 will not
- In fact, the solution will have the following form

The Second Best Contract



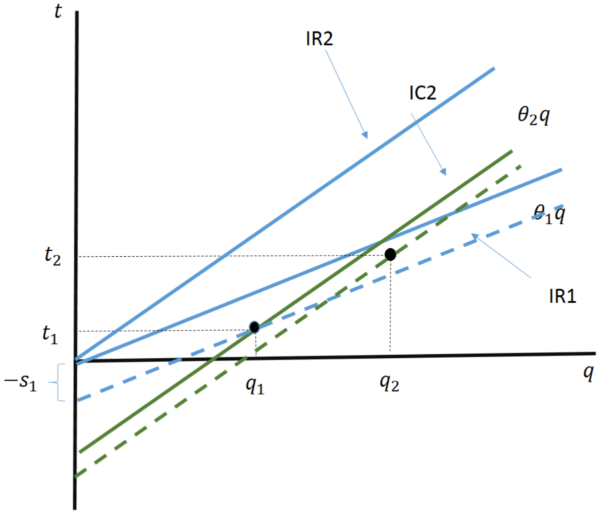
The Second Best Contract

- Specifically
  1. IR1 binds
  2. IC2 binds
  3. $q_2 \geq q_1$
  4. IC1 and IR2 are slack
  5. Sophisticated consumers buy the first best quantity
- We will prove these 5 statements one at a time

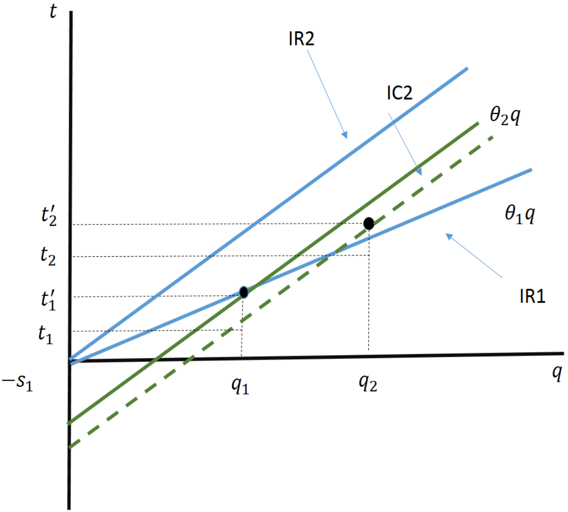
The Second Best Contract

- First: IR1 binds
  - This implies that
$$\theta_1 q_1 - t_1 = 0$$
  - Assume not
$$\theta_1 q_1 - t_1 > 0$$
  - We also know that
$$\theta_2 q_2 - t_2 \geq \theta_2 q_1 - t_1 > \theta_1 q_1 - t_1$$
  - So if IR1 does not bind, then nor does IR2
  - Could increase t_1 and t_2 by some amount ε
  - IR still bind, no effect on IC

The Second Best Contract



The Second Best Contract



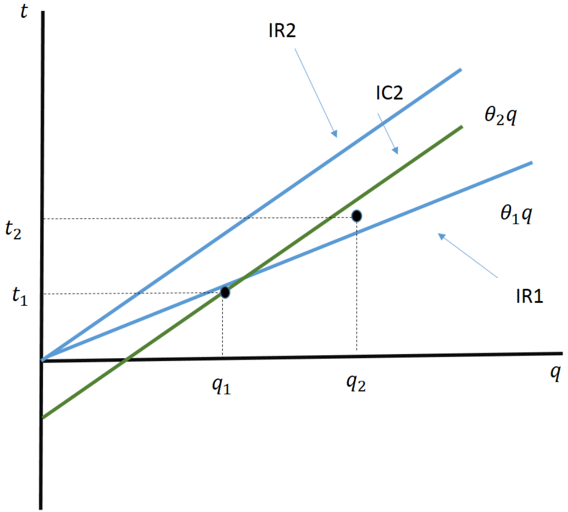
The Second Best Contract

- Second: IC2 binds
  - Assume not, then

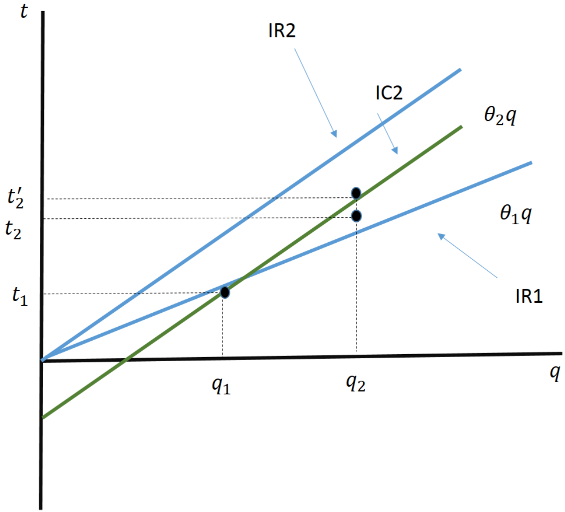
$$\theta_2 q_2 - t_2 > \theta_2 q_1 - t_1 \geq \theta_1 q_1 - t_1 = 0$$

  - But this means that IR2 also does not bind
  - Increase t_2 and IC2 and IR2 will continue to hold
  - (as will IC1 and IR1)

The Second Best Contract



The Second Best Contract



The Second Best Contract

- Third, $q_2 \geq q_1$
  - Add the two IC constraints

$$\begin{aligned} \theta_1 q_1 - t_1 &\geq \theta_1 q_2 - t_2 \\ \theta_2 q_2 - t_2 &\geq \theta_2 q_1 - t_1 \end{aligned}$$

  - Implies

$$\theta_2 \left(q_2 - q_1\right) \geq \theta_1 \left(q_2 - q_1\right)$$

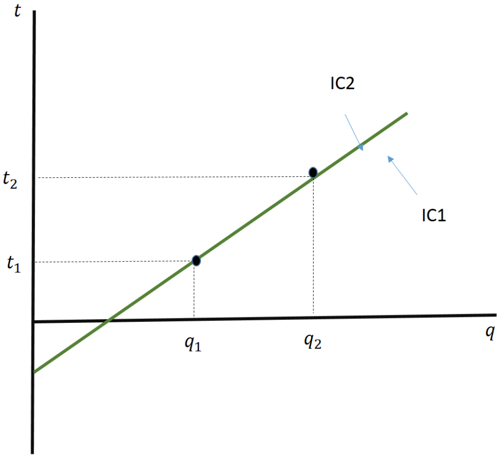
  - Which implies $(q_2 - q_1) \geq 0$ and $\theta_2 > \theta_1$

The Second Best Contract

- Fourth, IC1 is slack
  - We know that IC2 is active

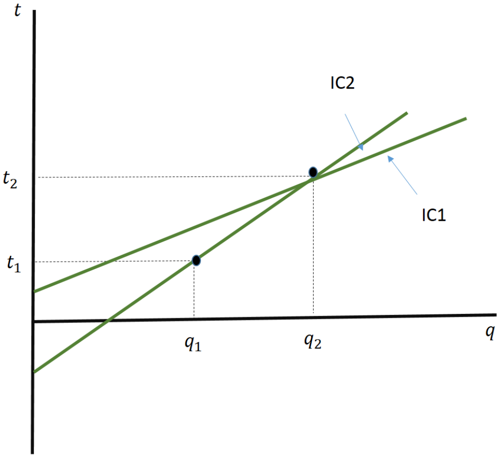
$$\begin{aligned}
\theta_2 q_2 - t_2 &= \theta_2 q_1 - t_1 \Rightarrow \\
t_2 - t_1 &= \theta_2 (q_2 - q_1) \\
&> \theta_1 (q_2 - q_1) \Rightarrow \\
\theta_1 q_1 - t_1 &> \theta_1 q_2 - t_2
\end{aligned}$$

The Second Best Contract



- If IC2 is active and $q_2 \geq q_1$....

The Second Best Contract



- If IC1 must hold

The Second Best Contract

- Proof of claim 1 shows that if IR1 and IC2 hold then IR2 also holds

The Second Best Contract

- Sophisticated consumers buy the first best quantity
    - i.e. $C'(q_2^*) = \theta_2$
- Assume not, so (for example) $C'(q_2) < \theta_2$
- Consider a new contract

$$\begin{aligned} q_1' &= q_1 \\ t_1' &= t_1 \\ q_2' &= q_2 + \varepsilon \\ t_2' &= t_2 + \varepsilon\theta_2 \end{aligned}$$

- Easy to see (check!) that this new contract satisfies all 4 constraints
- But, by Taylor expansion

$$t_2' - C(q_2') \simeq t_2 - C(q_2) + \varepsilon(\theta_2 - C'(q_2))$$

The Second Best Contract

- So we have the following conditions which characterize the optimum
    1. $q_2 = q_2^*$ where $C'(q_2^*) = \theta_2$
    2. $\theta_1 q_1 = t_1$
    3. $t_2 = \theta_1 q_1 + \theta_2 (q_2^* - q_1)$
- Three equations, 4 unknowns
- But we can substitute into the firm's optimization problem to get fourth unknown

The Second Best Contract

- Firm's objective function

$$
\begin{aligned}
& \max_{q_1,t_1,q_2,t_2} \pi_1[t_1 - C(q_1)] + \pi_2[t_2 - C(q_2)] \\
\Rightarrow \quad & \max_{q_1} \pi_1[\theta_1 q_1 - C(q_1)] + \pi_2[\theta_1 q_1 + \theta_2(q_2^* - q_1) - C(q_2^*)] \\
= \quad & \max_{q_1} \pi_1[\theta_1 q_1 - C(q_1)] - \pi_2 q_1 (\theta_2 - \theta_1) + \pi_1(q_2^* - C(q_2^*))
\end{aligned}
$$

- Last term is a constant, so can be ignored from the point of view of optimization, so

$$
\max_{q_1} \pi_1[\theta_1 q_1 - C(q_1)] - \pi_2 q_1 (\theta_2 - \theta_1)
$$

The Second Best Contract

- Note: Social surplus of type 1 agents is the sum of utility of the agent and profit

$$\begin{aligned} \sigma_1(q_1) &= \theta_1 q_1 - t_1 + t_1 - C(q_1) \\ &= \theta_1 q_1 - C(q_1) \end{aligned}$$

- Firm optimizes

$$\sigma_1(q_1) - \frac{\pi_2}{\pi_1} q_1 (\theta_2 - \theta_1)$$

- This is sometimes called virtual surplus
  - Social surplus of type 1
  - Adjustment necessary to keep type 2 honest

The Second Best Contract

- Gives first order conditions

$$C'(q_1) = \theta_1 - \frac{\pi_2}{\pi_1}(\theta_2 - \theta_1) < \theta_1$$

- q_1 chosen to set marginal cost less than θ_1
- Lower quality than the first best solution

The Second Best Contract

- Summarizing the second best contract
  - High type gets efficient allocation
  - Every type is indifferent between their contract and that of the type immediately below
  - All types but the lowest get positive surplus (informational rent)
  - All types but the highest get less than efficient allocation
  - Lowest type gets zero surplus

Corner Solution

- So far we have assumed that it is always optimal for the firm to sell to both the high type and the low type
- Is this always the case?

$$\max_{q_1} \pi_1[\theta_1 q_1 - C(q_1)] - \pi_2 q_1 (\theta_2 - \theta_1)$$

- No! If π_1 is low then it may not be worth it to sell to them, due to the distortion on the high type
- The above expression is **decreasing** in q_1
- Optimal contract excludes low type
- Just offer the high type their first best contract