G5212: Game Theory

Mark Dean

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Cheap Talk

- In the previous lecture honesty in signalling was ensured by costs
  - Different costs for different types meant that no-one had incentive to lie
- Today we will look at models of **cheap talk**
  - All types have the same (zero) cost of sending each message
- Can we have communication?
  - Obviously yes, if interests are perfectly aligned
  - Think of members of a bomb disposal squad!
- But we will show that we can also have communication if interests are partially aligned

Cheap Talk

- Will we be able to guarantee meaningful communication?
- No, we will never be able to rule out ‘babbling’ equilibria
  - Sender randomizes between signals
  - Receiver ignores what is sent
- Need further refinements to rule this out
  - e.g. lying costs
  - beyond the scope of this course
- But we can find equilibria in which communication takes place
- We will
  - Start with a simple, specific example in which we show how cheap talk can improve efficiency
  - Describe a more general model

A Simple Model

- N villagers
- Each has to choose between hunting or shirking
- Has an individual cost of hunting c_n drawn uniformly from $[0, 1 + \varepsilon]$
- Cost is private information
- If everyone hunts then each villager gets benefit 1
- Otherwise there is no benefit from hunting

A Simple Model

- Let S_n be the strategy of player n
  - 1 if hunt, 0 if shirk
- So payoff to player n is

$$\begin{aligned} 1 - c_n \text{ if } S_i &= 1 \text{ all } i \\ -c_n \text{ if } S_n &= 1 \text{ but } S_i = 0 \text{ for some } i \\ & 0 \text{ otherwise} \end{aligned}$$

A Simple Model

- First, let's think about this game with no communication
- Claim: Only equilibrium is one in which no one goes hunting
- First, note that is clearly an equilibrium
    - If no one else is hunting then clearly I do not want to hunt

A Simple Model

- Second, note that it is the only equilibrium
  - Let π be the equilibrium probability that one villager hunts
  - Payoff of hunting is π^{N-1}
  - Equilibrium is a cutoff rule
  - Hunt only if costs c_i are below π^{N-1}
  - Thus we have

$$\begin{aligned} \pi &= \frac{c}{1+\varepsilon} = \frac{\pi^{N-1}}{1+\varepsilon} \\ &\Rightarrow (1+\varepsilon) = \pi^{N-2} \end{aligned}$$

  - $\pi = 0$ only solution with $\pi \leq 1$

A Simple Model

- So now let's add some pre-play communication
  - Stage 1: Villagers announce 'yes' or 'no'
  - Stage 2: Each villager decides whether to hunt or not conditional on the announcements in stage 1
- Claim: the following is an equilibrium
  - In stage 1, report 'Yes' if $c_i \leq 1$
  - In stage 2, hunt if and only if everyone says 'Yes' in stage 1

A Simple Model

- Clearly this is an equilibrium in the second stage
  - Assume everyone else has voted yes
  - Taking the strategies of everyone else as given then everyone else will hunt
  - I would prefer to hunt as long as $c_i \leq 1$
  - If I voted yes in the first stage this must be the case
  - If one other person voted no, then there is no chance of success if I hunt - would rather not hunt

A Simple Model

- And at the first stage
    - If I have $c_i > 1$ cannot profit by deviating to "Yes"
    - If I have $c_i \leq 1$ cannot profit by deviating to "No"
- Notes
    - Babbling equilibrium still exists
    - "Yes" and "No" are purely conventions

Crawford-Sobel

- We will now have a look at the classic Crawford-Sobel cheap talk model
- This formalizes the idea that the amount of information which can be transmitted depends on how well aligned preferences are
- Uses a fairly stylized framework to do so
- Two agents
    - Sender: Observes a state of the world $m \in [0, 1]$
    - Sends a signal $n \in [0, 1]$ to a receiver
    - Receiver initially has a prior given by cdf μ
    - Updates it based on signal to $r(.|n)$
    - Takes action y

Crawford-Sobel

- Utilities given by
- For the sender

$$U^S(y, m)$$

  - Concave in y
  - maximum at $y = y^s(m)$ - sender's preferred action -which is increasing in m

- For the receiver

$$U^R(y, m)$$

  - Also concave in y
  - Maximized at $y^R(m) \neq y^S(m)$

Crawford-Sobel

- For example

$$\begin{aligned} U^S(y,m) &= -(y-m)^2 \\ U^R(y,m) &= -(y-m-a)^2 \end{aligned}$$

- so

$$\begin{aligned} y^S(m) &= m \\ y^R(m) &= m+a \end{aligned}$$

- $|y^S - y^R|$ measures the degree of disagreement

Solution

- Correct solution concept is weak Perfect Bayesian Equilibrium
  - Signal strategy by the sender $q^* : [0,1] \to [0,1]$ where $q^*(m)$ is the signal sent if the state of the world is m
  - Belief function r^* such that $r^*(.|n)$ is the beliefs formed upon receipt of signal n
  - Action strategy y^* where $y^*(n)$ is the action taken upon receipt of signal n

Solution

- Such that
  - Signal strategy is optimal given recipient's strategy

$$q^*(m) \in \arg \max_{n \in [0,1]} U^S(y^*(n), m)$$

  - Actions are optimal given beliefs

$$y^*(n) \in \arg\max_{y} \int_m U^R(y, m) r^*(m|n) dm$$

  - Beliefs are formed using Bayes' rule where possible

Partition Equilibria

- We will focus on partition equilibria
  - State space is divided into p subintervals denoted $[m_{i-1}, m_i]$ with $m_0 = 0$ and $m_p = 1$
- Signal sent depends only on the subinterval
  - sender sends only $n_1 < n_2 < ...n_p$

$$q^*(m) = n_i \text{ for any } q \in [m_{i-1}, m_i]$$

Theorem (Crawford and Sobel)

For any cheap talk game there exists an integer N such that, for any $p \leq N$, there is a partition equilibrium of the game with p partitions

Partition Equilibrium

- We will now construct an example of a partition equilibrium for the quadratic case

$$\begin{aligned} U^S(y,m) &= -(y-m)^2 \\ U^R(y,m) &= -(y-m-a)^2 \end{aligned}$$

- With μ uniform
- In particular we will construct the partition equilibrium for $p = 3$

Partition Equilibrium

- First, let's think of the best response of the recipient
- How should they respond upon receiving signal n_i?
- Remember that in equilibrium they 'know' the strategy of the sender
- So they know upon receiving n_i that m is uniformly distributed between m_{i-1} and m_i
  - $r^*(m|n_i) = U[m_{i-1}, m_i]$
- Objective function is therefore

$$\int_{m_{i-1}}^{m_i} -(y-m-a)^2 \left(\frac{1}{m_i - m_{i-1}}\right) dm$$

Partition Equilibrium

- Taking derivatives with respect to y gives

$$\int_{m_{i-1}}^{m_i} -2(y-m-a)\left(\frac{1}{m_i - m_{i-1}}\right) dm = 0$$

$$\Rightarrow \quad \left[-2\left(\left((y-a)m - \frac{m^2}{2}\right)\right)\right]_{m_{i-1}}^{m_i} = 0$$

$$\Rightarrow \quad (y-a)(m_i - m_{i-1}) - \left(\frac{m_i^2 - m_{i-1}^2}{2}\right) = 0$$

$$\Rightarrow \quad (y-a)(m_i - m_{i-1}) = \frac{(m_i - m_{i-1})(m_i + m_{i-1})}{2}$$

$$\Rightarrow \quad y^*(n_i) = \frac{m_i + m_{i-1}}{2} + a$$

Partition Equilibrium

- What about the sender?
- They have to prefer to send message n_i to any other message for any m in $[m_{i-1}, m_i]$

$$U^S(y_i, m) \geq U^S(y_j, m) \text{ for every } m \in [m_{i-1}, m_i]$$

- It is sufficient to check that at the boundary point m_i the sender is indifferent between sending signals n_i and n_{i+1}
  - This means that for $m > m_i$ then n_{i+1} will be strictly preferred
  - For $m < m_i$, n_i is strictly preferred

Partition Equilibrium

- So the condition becomes

$$U^S\left(y^*(n_i), m_i\right) = U^S\left(y^*(n_{i+1}), m_i\right)$$

- Plugging in

$$U^S(y,m) = -(y-m)^2$$

and

$$y^*(n_i) = \frac{m_i + m_{i-1}}{2} + a$$

gives

$$\left(\frac{m_{i-1} + m_i}{2} + a - m_i\right)^2 = \left(\frac{m_{i+1} + m_i}{2} + a - m_i\right)^2$$

Partition Equilibrium

$$\left(\frac{m_{i-1}+m_i}{2}+a-m_i\right)^2=\left(\frac{m_{i+1}+m_i}{2}+a-m_i\right)^2$$

- As $\frac{m_{i-1}+m_i}{2}<\frac{m_{i+1}+m_i}{2}$ this requires LHS to be negative and RHS to be positive

$$\begin{aligned}\frac{m_{i-1}+m_i}{2}+a-m_i &= m_i-a-\frac{m_{i+1}+m_i}{2}\\ &\Rightarrow m_{i+1}=2m_i-m_{i-1}-4a\end{aligned}$$

Partition Equilibrium

- This is a difference equation.
    - Break out the maths notes![1]
- Solution is of the form

$$m_i = \lambda i^2 + \mu i + v$$

- Plugging in to

$$\begin{aligned} m_3 &= 2m_2 - m_1 - 4a \\ &\Rightarrow 9\lambda + 3\mu + v \\ &= 2(4\lambda + 2\mu + v) \\ &\quad -(\lambda + \mu - v) \\ &\quad -4a \end{aligned}$$

- so $\lambda = -2a$

[1] https://www.cl.cam.ac.uk/teaching/2003/Probability/prob07.pdf page 7.8

Partition Equilibrium

- Also, we know that $m_0 = 0$
- This implies that

$$\begin{aligned} m_2 &= 2m_1 - 4a \\ &\Rightarrow 4\lambda + 2\mu + v \\ &= 2\lambda + 2\mu + 2v - 4a \\ &\Rightarrow -8a + v \\ &= -4a + 2v - 4a \\ &\Rightarrow v = 0 \end{aligned}$$

Partition Equilibrium

- Finally we know that $m_p = 1$
- This implies that

$$\begin{aligned} m_i &= \lambda i^2 + \mu i + v \\ &\Rightarrow 1 = -2ap^2 + \mu p \\ &\Rightarrow \mu = \frac{1}{p} + 2ap \end{aligned}$$

- And so the general solution is

$$m_i = -2ai^2 + \left(\frac{1}{p} + 2ap\right) i$$

Partition Equilibrium

- And in the specific case of $p = 3$

$$
\begin{aligned}
m_0 &= 0 \\
m_1 &= \frac{1}{3} + 4a \\
m_2 &= \frac{2}{3} + 4a \\
m_3 &= 0
\end{aligned}
$$

Partition Equilibrium

- How many partitions can we support?
- Well, for the solution to be valid, we need m_i to be increasing
- Rewriting

$$m_{i+1} = 2m_i - m_{i-1} - 4a$$

as

$$m_{i+1} - m_i = m_i - m_{i=1} - 4a$$

we get

$$\begin{aligned}
m_2 - m_1 &= m_1 - m_0 - 4a \\
m_3 - m_2 &= m_1 - m_0 - 8a \\
&\vdots \\
m_p - m_{p-1} &= m_1 - m_0 - (p-1)4a
\end{aligned}$$

Partition Equilibrium

- So for the sequence to be increasing we need

$$m_1 - m_0 > (p-1)4a$$

- Or, plugging back in

$$\frac{1}{p} + 2a(1-p) > 0$$

- As $\lim_{p\to\infty} = -\infty$, this defines the maximal possible p that can be supported
- Decreasing in a
- Notice that the actual nature of the signal is meaningless
- Could use name of football teams instead!