G5212: Game Theory

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In the previous lecture honesty in signalling was ensured by costs

- Different costs for different types meant that no-one had incentive to lie

Today we will look at models of **cheap talk**

- All types have the same (zero) cost of sending each message

Can we have communication?

- Obviously yes, if interests are perfectly aligned
  - Think of members of a bomb disposal squad!

But we will show that we can also have communication if interests are partially aligned
Will we be able to guarantee meaningful communication?

No, we will never be able to rule out ‘babbling’ equilibria

- Sender randomizes between signals
- Receiver ignores what is sent

Need further refinements to rule this out

- e.g. lying costs
- beyond the scope of this course

But we can find equilibria in which communication takes place

We will

- Start with a simple, specific example in which we show how cheap talk can improve efficiency
- Describe a more general model
A Simple Model

- $N$ villagers
- Each has to choose between hunting or shirking
- Has an individual cost of hunting $c_n$ drawn uniformly from $[0, 1 + \varepsilon]$
- Cost is private information
- If everyone hunts then each villager gets benefit 1
- Otherwise there is no benefit from hunting
A Simple Model

- Let $S_n$ be the strategy of player $n$
  - 1 if hunt, 0 if shirk
- So payoff to player $n$ is

\[
1 - c_n \text{ if } S_i = 1 \text{ all } i \\
-c_n \text{ if } S_n = 1 \text{ but } S_i = 0 \text{ for some } i \\
0 \text{ otherwise}
\]
First, let’s think about this game with no communication.

Claim: Only equilibrium is one in which no one goes hunting.

First, note that is clearly an equilibrium:

If no one else is hunting then clearly I do not want to hunt.
Second, note that it is the only equilibrium

- Let $\pi$ be the equilibrium probability that one villager hunts
- Payoff of hunting is $\pi^{N-1}$
- Equilibrium is a cutoff rule
- Hunt only if costs $c_i$ are below $\pi^{N-1}$
- Thus we have

$$
\pi = \frac{c}{1 + \varepsilon} = \frac{\pi^{N-1}}{1 + \varepsilon}
$$

$$
\Rightarrow (1 + \varepsilon) = \pi^{N-2}
$$

- $\pi = 0$ only solution with $\pi \leq 1$
So now let’s add some pre-play communication

Stage 1: Villagers announce ’yes’ or ‘no’
Stage 2: Each villager decides whether to hunt or not conditional on the announcements in stage 1

Claim: the following is an equilibrium

In stage 1, report ‘Yes’ if $c_i \leq 1$
In stage 2, hunt if and only if everyone says ‘Yes’ in stage 1
Clearly this is an equilibrium in the second stage

- Assume everyone else has voted yes
- Taking the strategies of everyone else as given then everyone else will hunt
- I would prefer to hunt as long as $c_i \leq 1$
- If I voted yes in the first stage this must be the case
- If one other person voted no, then there is no chance of success if I hunt - would rather not hunt
A Simple Model

- And at the first stage
  - If I have $c_i > 1$ cannot profit by deviating to "Yes"
  - If I have $c_i \leq 1$ cannot profit by deviating to "No"

- Notes
  - Babbling equilibrium still exists
  - "Yes" and "No" are purely conventions
We will now have a look at the classic Crawford-Sobel cheap talk model.

This formalizes the idea that the amount of information which can be transmitted depends on how well aligned preferences are.

It uses a fairly stylized framework to do so.

Two agents:
- **Sender**: Observes a state of the world \( m \in [0, 1] \)
- Sends a signal \( n \in [0, 1] \) to a receiver.
- **Receiver** initially has a prior given by cdf \( \mu \).
- Updates it based on signal to \( r(.|n) \).
- Takes action \( y \).
Utilities given by

For the sender
\[ U^S(y, m) \]
- Concave in \( y \)
- Maximum at \( y = y^s(m) \) - sender’s preferred action - which is increasing in \( m \)

For the receiver
\[ U^R(y, m) \]
- Also concave in \( y \)
- Maximized at \( y^R(m) \neq y^S(m) \)
For example

\[ U^S(y, m) = -(y - m)^2 \]
\[ U^R(y, m) = -(y - m - a)^2 \]

so

\[ y^S(m) = m \]
\[ y^R(m) = m + a \]

\[ |y^S - y^R| \] measures the degree of disagreement
Solution

- Correct solution concept is weak Perfect Bayesian Equilibrium
  - Signal strategy by the sender \( q^*: [0, 1] \rightarrow [0, 1] \) where \( q^*(m) \) is the signal sent if the state of the world is \( m \)
  - Belief function \( r^* \) such that \( r^*(.|n) \) is the beliefs formed upon receipt of signal \( n \)
  - Action strategy \( y^* \) where \( y^*(n) \) is the action taken upon receipt of signal \( n \)
Solution

- Such that
  - Signal strategy is optimal given recipient’s strategy
    \[ q^*(m) \in \arg\max_{n \in [0,1]} U^S(y^*(n), m) \]
  - Actions are optimal given beliefs
    \[ y^*(n) \in \arg\max_{y} \int_m U^R(y, m)r^*(m|n)dm \]
  - Beliefs are formed using Bayes’ rule where possible
Partition Equilibria

- We will focus on partition equilibria
  - State space is divided into $p$ subintervals denoted $[m_{i-1}, m_i]$ with $m_0 = 0$ and $m_p = 1$
  - Signal sent depends only on the subinterval
    - sender sends only $n_1 < n_2 < \ldots n_p$
    \[ q^*(m) = n_i \text{ for any } q \in [m_{i-1}, m_i] \]

Theorem (Crawford and Sobel)

*For any cheap talk game there exists an integer $N$ such that, for any $p \leq N$, there is a partition equilibrium of the game with $p$ partitions*
We will now construct an example of a partition equilibrium for the quadratic case

\[ U^S(y, m) = -(y - m)^2 \]
\[ U^R(y, m) = -(y - m - a)^2 \]

With \( \mu \) uniform

In particular we will construct the partition equilibrium for \( p = 3 \)
First, let’s think of the best response of the recipient

How should they respond upon receiving signal \( n_i \)?

Remember that in equilibrium they ‘know’ the strategy of the sender

So they know upon receiving \( n_i \) that \( m \) is uniformly distributed between \( m_{i-1} \) and \( m_i \)

\[ r^*(m|n_i) = U[m_{i-1}, m_i] \]

Objective function is therefore

\[
\int_{m_{i-1}}^{m_i} -(y - m - a)^2 \left( \frac{1}{m_i - m_{i-1}} \right) \, dm
\]
Taking derivatives with respect to $y$ gives

$$\int_{m_{i-1}}^{m_i} -2(y - m - a) \left( \frac{1}{m_i - m_{i-1}} \right) \, dm = 0$$

$$\Rightarrow \left[ -2 \left( (y - a)m - \frac{m^2}{2} \right) \right]_{m_{i-1}}^{m_i} = 0$$

$$\Rightarrow (y - a)(m_i - m_{i-1}) - \left( \frac{m_i^2 - m_{i-1}^2}{2} \right) = 0$$

$$\Rightarrow (y - a)(m_i - m_{i-1}) = \frac{(m_i - m_{i-1})(m_i + m_{i-1})}{2}$$

$$\Rightarrow y^\ast(n_i) = \frac{m_i + m_{i-1}}{2} + a$$
What about the sender?

They have to prefer to send message $n_i$ to any other message for any $m$ in $[m_{i-1}, m_i]$

$$U_S(y_i, m) \geq U_S(y_j, m) \text{ for every } m \in [m_{i-1}, m_i]$$

It is sufficient to check that at the boundary point $m_i$ the sender is indifferent between sending signals $n_i$ and $n_{i+1}$

- This means that for $m > m_i$ then $n_{i+1}$ will be strictly preferred
- For $m < m_i$, $n_i$ is strictly preferred
So the condition becomes

$$U^S(y^*(n_i), m_i) = U^S(y^*(n_{i+1}), m_i)$$

Plugging in

$$U^S(y, m) = -(y - m)^2$$

and

$$y^*(n_i) = \frac{m_i + m_{i-1}}{2} + a$$

gives

$$\left(\frac{m_{i-1} + m_i}{2} + a - m_i\right)^2 = \left(\frac{m_{i+1} + m_i}{2} + a - m_i\right)^2$$
\[
\left( \frac{m_{i-1} + m_i}{2} + a - m_i \right)^2 = \left( \frac{m_{i+1} + m_i}{2} + a - m_i \right)^2
\]

As \( \frac{m_{i-1} + m_i}{2} < \frac{m_{i+1} + m_i}{2} \) this requires LHS to be negative and RHS to be positive

\[
\frac{m_{i-1} + m_i}{2} + a - m_i = m_i - a - \frac{m_{i+1} + m_i}{2}
\]
\[
\Rightarrow m_{i+1} = 2m_i - m_{i-1} - 4a
\]
Partition Equilibrium

- This is a difference equation.
  - Break out the maths notes!\(^1\)
- Solution is of the form
  \[ m_i = \lambda i^2 + \mu i + v \]
- Plugging in to
  \[ m_3 = 2m_2 - m_1 - 4a \]
  \[ \Rightarrow 9\lambda + 3\mu + v \]
  \[ = 2(4\lambda + 2\mu + v) \]
  \[ - (\lambda + \mu - v) \]
  \[ - 4a \]

- so \( \lambda = -2a \)

\(^1\)https://www.cl.cam.ac.uk/teaching/2003/Probability/prob07.pdf page 7.8
Also, we know that $m_0 = 0$

This implies that

$$m_2 = 2m_1 - 4a$$

$$\Rightarrow 4\lambda + 2\mu + v$$

$$= 2\lambda + 2\mu + 2v - 4a$$

$$\Rightarrow -8a + v$$

$$= -4a + 2v - 4a$$

$$\Rightarrow v = 0$$
Finally we know that $m_p = 1$

This implies that

$$m_i = \lambda i^2 + \mu i + v$$

$$\Rightarrow 1 = -2ap^2 + \mu p$$

$$\Rightarrow \mu = \frac{1}{p} + 2ap$$

And so the general solution is

$$m_i = -2ai^2 + \left(\frac{1}{p} + 2ap\right)i$$
And in the specific case of $p = 3$

\[
\begin{align*}
    m_0 & = 0 \\
    m_1 & = \frac{1}{3} + 4a \\
    m_2 & = \frac{2}{3} + 4a \\
    m_3 & = 0
\end{align*}
\]
How many partitions can we support?

Well, for the solution to be valid, we need $m_i$ to be increasing

Rewriting

$$m_{i+1} = 2m_i - m_{i-1} - 4a$$

as

$$m_{i+1} - m_i = m_i - m_{i-1} - 4a$$

we get

$$m_2 - m_1 = m_1 - m_0 - 4a$$

$$m_3 - m_2 = m_1 - m_0 - 8a$$

$$\vdots$$

$$m_p - m_{p-1} = m_1 - m_0 - (p - 1)4a$$
So for the sequence to be increasing we need

\[ m_1 - m_0 > (p - 1)4a \]

Or, plugging back in

\[ \frac{1}{p} + 2a(1 - p) > 0 \]

As \( \lim_{p \to \infty} = -\infty \), this defines the maximal possible \( p \) that can be supported.

Decreasing in \( a \)

Notice that the actual nature of the signal is meaningless

Could use name of football teams instead!