Introduction

What do your preferences look like?

The Story So Far....

1. In Monday's lecture we described how we would model consumer choice as constrained optimization:
   1. CHOOSE a consumption bundle
   2. IN ORDER TO MAXIMIZE preferences
   3. SUBJECT TO the budget constraint
   Consumption bundles and budget constrains were dealt with fairly thoroughly
   Preferences may still seem a bit mysterious
   Today, we will attempt to de-mystify

Today's Aims

1. Describe how to represent preferences on our handy graphs
   - Indifference Curves
2. Describe how to represent preferences using maths
   - Utility functions
3. Describe “standard” preferences
   - Monotonic
   - Convex
4. Introduce some other handy classes of preferences
   - Perfect substitutes
   - Perfect compliments
   - Cobb Douglas
   Reminder: Varian Ch. 3 & 4, Feldman and Serrano Ch 2

Going from Three Dimensions to Two

- Think back to our nice, simple example with two goods
- Made it nice and easy to graph the consumption bundles and budget constraints

Indifference Curves

Or how to see your preferences
Going from Three Dimensions to Two

- Think back to our nice, simple example with two goods
- Made it nice and easy to graph the consumption bundles and budget constraints

\[
\text{Budget constraint is } \frac{m}{p_2}x_1 + \frac{m}{p_1} = m.
\]

Going from Three Dimensions to Two

- Unfortunately, we now have three pieces of information associated with each bundle
  - Amount of good 1
  - Amount of good 2
  - Whether this bundle is preferred or not to others
- How can we represent this information?
  - One way would be to use three dimensions:
    - Bundles that are more preferred are placed higher on dimension 3 than those which are less preferred

Going from Three Dimensions to Two

- While this would work, graphing in three dimensions is a pain
- As we will see, we can get a lot more intuition if we can work in two dimensions
- So how can we go from three dimensions to two?
  - Luckily, cartographers have the answer

Going from Three Dimensions to Two

- On a map, contour lines link areas of equal height
- We will use indifference curves which link areas of equal preference
Indifference Curves

- A simple example
- 2 goods:
  - Apples
  - Oranges
- Suppose our agent likes both
- Take some bundle $x'$
- An indifference curve links all bundles of goods which are indifferent to $x'$
- For example, say that $x' \sim x'' \sim x'''$

Graphically

3 curves
All bundles in $I_1$ are strictly preferred to all in $I_2$.
All bundles in $I_2$ are strictly preferred to all in $I_3$. 
**Indifference Curves**

- Indifference Curves represent the set of bundles weakly preferred to a point $x$.
  \[ I(x) \]

- The set of bundles strictly preferred to $x$ is represented by a dashed line.
  \[ I(x) \]

**Can Indifference Curves Intersect?**

- Take two curves $I_1$ and $I_2$.
- Suppose $z < y$ so these are different curves.
- Can they intersect?

**The Marginal Rate of Substitution**

- One crucial thing we want to know about preferences:
  "If I take away one apple, how many oranges would I have to give you to keep you indifferent?"

- Or more accurately, at what rate do I have to change oranges for apples to keep you indifferent?

- This is the **marginal rate of substitution** (MRS) between apples and oranges.

- Mathematically,
  \[
  MRS(x_u, x_o) = -\lim_{\Delta x_o \to 0} \frac{\Delta x_u}{\Delta x_o}
  \]

  Such that $(x_u, x_o) = (x_u + \Delta x_u, x_o + \Delta x_o)$

**Why is this of interest?**

- At this stage you can take my word for it...

- Or you can think about the following:
  "If the rate at which I am willing to trade off apples to oranges is higher than the relative price of apples and oranges, am I maximizing my preferences?"

**WARNING:** Sometimes MRS is defined without the minus sign.

- There is no consensus about which is correct.

- I will accept either definition from you, as long as you are consistent!
The Marginal Rate of Substitution

- Claim: It is very easy to see the marginal rate of substitution from the indifference curve.
- The MRS at a particular point is the negative of the slope of the indifference curve at that point.

Marginal Rate of Substitution

- Slope of indifference curve at \((x_1, x_2)\) is the MRS at this point.

Preferences and Utility

Or: Your preferences in numbers

Where are the Numbers?

- So far we have represented the objectives in our consumer’s problem with preferences and weird symbols.
- I bet you are crying out to get away from these symbols and use some proper, god fearing numbers to represent what the consumer wants to maximize.
- This would allow us to do lots of cool things.
- For example take derivatives.
- Moreover, you have probably heard of the concept of a ‘utility function’.
- Reports how ‘happy’ a particular bundle of goods makes someone.
- So why can’t we work with utility functions?

Where are the Numbers?

- Okay, so we are going to work with utility functions.
- But first I want you to understand how utility functions are used in economics.
- This is slightly counterintuitive...
- ...but useful for you to understand what is going on.
- For many people this is the most confusing bit of the course, so hold on!
In the beginning....

- Back in the days of Marshall (1890's) utility was considered to be a real, measurable, cardinal scale
- Lurking in people’s brains was the ‘happiness’, or utility, associated with different bundles of goods
- People made choices in order to maximize this happiness
- The nature of utility could be derived from first principles
  - Calories
  - Water
  - Warmth
  - Etc.

Three problems with this approach
1. What evidence did we have that there was utility lurking in the brain
   - No way to directly measure it
2. All attempts to derive utility from first principles (and so explain choice) failed
3. What did it mean for the utility of bundle x to be twice that of bundle y?

20th Century: Preferences!

- This lead to a change of approach in 20th century
- Preferences were taken as the primitive thing that people maximized
- Why? What is the big advantage of preferences over utility?
- We can plausibly measure preferences in a way we cannot measure utility
- How?
  - One good way would be through choice
    - x is preferred to y if x is chosen over y
  - Side note: Neuroscientists and neuro-economists are trying to take us back to the days of Marshall

Utility from Preferences

- Despite this realization, it is still useful to be able to work with utility functions
  - For one thing, we can take derivatives
- So the question became whether we can build a utility function from preferences
- Let’s start with a set of preferences \( ≿ \) on different bundles of goods
- Can we build a utility function \( u \) that represents (or contains the same information as) \( ≿ \)
  - I.e. we can find a way of assigning utility numbers to bundles such that, for any two bundles x and y
    \[ x ≿ y \text{ if and only if } u(x) ≥ u(y) \]

Utility from Preferences

- If we can do this, then we can ‘pretend’ that the consumer is maximizing utility
- Maximizing preferences: choosing x such that \( x ≿ y \) for all available y
- Maximizing utility: choosing x such that \( u(x) ≥ u(y) \) for available y
- These are the same thing if the utility function represents the preferences, i.e.
  \[ x ≿ y \text{ if and only if } u(x) ≥ u(y) \]

When do we have a utility function?

- So, when does a preference relation allow a utility representation?
- Answer: as long as it is well-behaved!
  - Complete
  - Transitive
  - Reflexive
- (Also, if the set of available options is not countable, we need continuity, but don’t worry so much about this)
When do we have a utility function?

- If preferences are not well behaved, will there be a utility representation?
- No!
- For example: failure of transitivity
  \[ x \preceq y, y \preceq z \text{ but NOT } x \preceq z \]
- Say \( u \) represents these preferences
  \[ x \preceq y \implies u(x) \geq u(y) \]
  \[ y \preceq z \implies u(y) \geq u(z) \]
  By the power of maths, this implies \( u(x) \geq u(z) \)
  But if \( u \) represents the preferences this would imply \( x \preceq z \)
  Contradiction
- A preference relation allows a utility representation if and only if it is well behaved

What do Utility Numbers Mean?

- Utility numbers only have ordinal meaning
  - The ordering of the numbers matters
- They do not have cardinal meaning
  - The difference between the numbers does not matter
- This is another way of saying that many utility functions represent the same preferences
- An obvious question: can we determine the relationship between utility functions that represent the same preferences?

Utility Functions & Indifference Curves

- We now have two ways of representing preferences
  - Indifference curves
  - Utility functions
- What is the relationship between them?
  - An indifference curve links equally preferred bundles
  - Equal preference \( \iff \) same utility level.
- Therefore, all bundles in an indifference curve have the same utility level.
- To each indifference curve we have one utility level

What do Utility Numbers Mean?

- Theorem: Take two utility functions \( u \) and \( v \). They both represent the same preferences if and only if there is a strictly increasing function \( f \) such that
  \[ v(x) = f(u(x)) \]
  for all \( x \)
- For example say \( u \) represents preferences and \( v(x) = 2u(x) + 3 \)
  \[ x \preceq y \text{ if and only if } u(x) \geq u(y) \]
  \[ 2u(x) + 3 \geq 2u(y) + 3 \text{ if and only if } v(x) \geq v(y) \]
Utility Functions & Indifference Curves

Plotting the utility on the vertical axis

\[
\begin{align*}
U(2,3) &= 6 \\
U(2,2) &= 4 \\
U(4,1) &= 4
\end{align*}
\]

Graphically

From utility to indifference curves

- Suppose we want to draw the indifference curves for a particular utility function \( u(x_1, x_2) = x_1 x_2 \)
- Indifference curves are elements with the same utility
- So? Just set the utility equal to \( k \)
  \[
  x_1 x_2 = k \implies x_2 = \frac{k}{x_1}
  \]
- This is the equation for the indifference curve for various \( k \)

An Intelligence test

- Another example
- Consider \( U(x_1, x_2) = x_1^2 x_2 \)
  
  - What do indifference curves look like?
  - We can proceed as before

An Intelligence test

- Another example
- Consider \( U(x_1, x_2) = x_1^2 x_2 \)
  
  - What do indifference curves look like?
  - We can proceed as before
  - Or: we can smart it out
    - Look: \( U(x_1, x_2) = x_1 x_2^2 = (x_1 x_2)^2 \)
    - But: \( f(y) = y^2 \) is a strictly increasing function
    - Thus: These are the same preferences as \( U(x_1, x_2) = x_1 x_2 \)
Marginal Utilities and The Marginal Rate of Substitution

- Recall that I claimed that we would be interested in the Marginal Rate of Substitution
- Slope of the indifference curve
- Rate at which you would change one good for another while remaining indifferent
- It will be very handy to know how to calculate the MRS from utility functions
- To do so we need to introduce the idea of Marginal Utility

Marginal Utility and MRS

- So what is the relationship between marginal utility and MRS?
- Recall that the MRS is the negative of the slope of the indifference curve
- And that the indifference curve is defined by $u(x_1, x_2) = k$ for some $k$
- In order to get the slope, take the total derivative
  \[ \frac{du}{dx_1}dx_1 + \frac{du}{dx_2}dx_2 = 0 \]
  Which implies
  \[ dx_2 = -\frac{\frac{du}{dx_1}}{\frac{du}{dx_2}}dx_1 \]
- Marginal Utility: the rate at which utility changes with the quantity of one good, keeping the other constant
  \[ \frac{\partial u}{\partial x_1} \]

Marginal Utility

- Marginal Utility: the rate at which utility changes with the quantity of one good, keeping the other constant
- The partial derivative of the utility function
  \[ MU_1 = \frac{\partial u(x)}{\partial x_1} \]
- So, for example, if $u(x_1, x_2) = x_1^2x_2$ then
  \[ MU_1 = \frac{\partial u(x_1, x_2)}{\partial x_1} = \frac{1}{2}x_1^2 \]
  \[ MU_2 = \frac{\partial u(x_1, x_2)}{\partial x_2} = 2x_1x_2 \]

Graphically

- $\text{MRS}(1,8) = \frac{8}{1} = 8$
- $\text{MRS}(6,6) = \frac{6}{6} = 1$

Are Marginal Utilities and MRS Meaningful?

- Recall that utility function is not unique
- I can “transform” it and still represent the same preferences
- But notice: marginal utility depends on the utility used
- Thus, if I transform the utility, I will change the marginal utility as well
- This means marginal utility has little behavioral content
- Don’t read too much into it!
- E.g., if I double the utility I double also the marginal utility but behaviorally identical!
- This is a subtle point: make sure you understand!
Monotonic Transformations &
Marginal Rates-of-Substitution

What about the MRS?
For \( U(x_1, x_2) = x_1x_2 \) the MRS = \( \frac{x_2}{x_1} \).
Create \( V = U^2 \); i.e., \( V(x_1, x_2) = x_1^2x_2^2 \). What is the MRS for \( V \)?
\[
\text{MRS} = \frac{\frac{dV}{dx_1}}{\frac{dV}{dx_2}} = \frac{2x_1^2x_2}{2x_1x_2^2} = \frac{x_2}{x_1}
\]
which is the same as the MRS for \( U \).

Monotonic Transformations &
Marginal Rates-of-Substitution

More generally, if \( V = f(U) \) where \( f \) is a strictly increasing function,
then
\[
\text{MRS} = \frac{\frac{du}{dx_1}}{\frac{du}{dx_2}} = \frac{f'(u(x)) \cdot \frac{du}{dx_1}}{f'(u(x)) \cdot \frac{du}{dx_2}}
\]
MRS unchanged by a positive monotonic transformation.

Summary

Today we have described two ways of representing preferences:
- Indifference curves (and the associated concept of MRS)
- Utility functions

Described the (somewhat confusing) relationship between preferences and utility:
- Utility used to represent preferences
- Are not unique — many utility functions represent the same preferences (and lead to the same choices)

Shown how to calculate MRS from utility functions:
- Marginal utility depends on the precise form on utility used
- MRS does not