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Intermediate Microeconomics W3211

Lecture 4:
Solving the Consumer's Problem

Columbia University, Spring 2016

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Introduction

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The Story So Far....

- We have now (exhaustively) described the consumer's problem

1. **CHOOSE a consumption bundle**
2. **IN ORDER TO MAXIMIZE preferences**
3. **SUBJECT TO the budget constraint**

- Now it is time to solve it!

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Today's Aims

1. Use pictures to think heuristically about how to solve the consumer's problem
   - Varian Ch. 5, Feldman and Serrano Ch 3
2. Use maths to turn this intuition into a solution method
   - It will be useful to review the materiel on first order conditions, Lagrangians etc
     - From your calculus class
     - Varian Ch. 5 appendix, Feldman and Serrano Ch. 3 appendix

A First Step in Solving The Consumer's Problem

Or "the single most important piece of the course"

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Solving the Consumer's Problem

- We now have all the pieces of the consumer's optimization problem

1. **CHOOSE a consumption bundle**
2. **IN ORDER TO MAXIMIZE preferences**
3. **SUBJECT TO the budget constraint**

- And we know how to represent preferences

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Solving the Consumer’s Problem

- The next lecture and a half will be concerned with how we solve the consumer’s problems
- This is the single most important part of the course
- We will be using the mathematics of **constrained optimization**
- This will turn up again and again throughout the course
  - If you get very comfortable with them now, this will really pay dividends
- You will only get better through practice!

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Solving the Consumer’s Problem

- To begin with, let’s draw some graphs
- See if we can solve this problem using raw brain power

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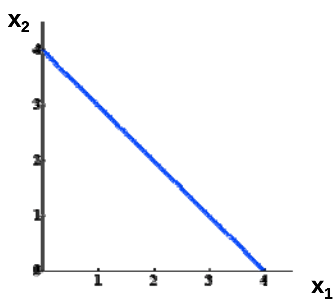
Rational Constrained Choice



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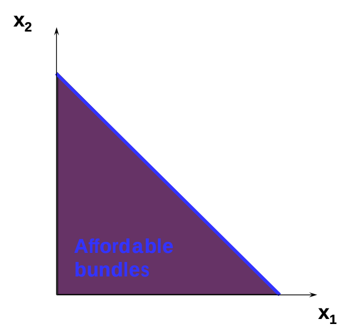
Rational Constrained Choice



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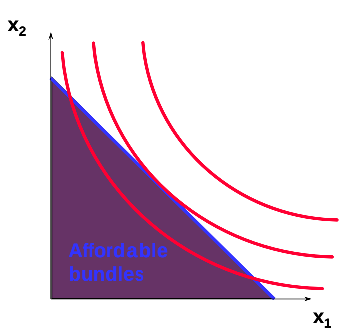
Rational Constrained Choice



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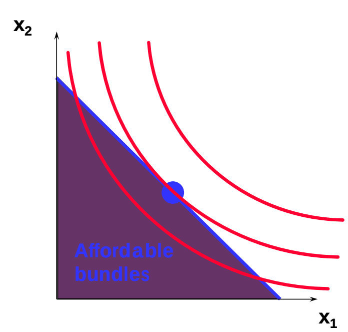

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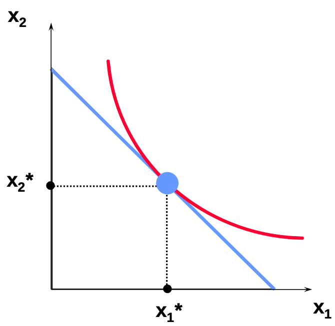

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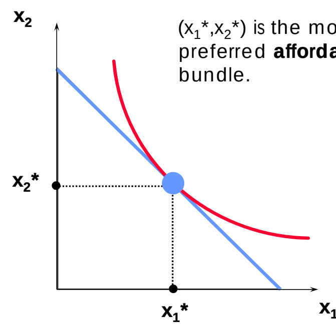
Rational Constrained Choice



(x_1^*, x_2^*) is the most preferred **affordable** bundle.

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Solving the Consumer's Problem

- What are the properties of this optimal point?

1. All the money is spent
   - i.e. the budget line holds at equality

$$p_1x_1 + p_2x_2 = y$$

2. The slopes of the indifference curve and the budget line are the same
   - i.e. the Marginal Rate of Substitution equals the ratio of prices
   - This is the **tangency condition**

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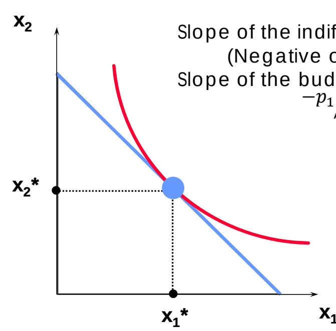
Rational Constrained Choice



Slope of the indifference curve:
(Negative of the) MRS
Slope of the budget line:
$-p_1/p_2$

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Solving the Consumer's Problem

- What is the intuition for this?

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Solving the Consumer's Problem

- What is the intuition for this?

1. Spending all the money:
   - If not, could get more of either good
     - Would make the consumer better off
     - Assuming preferences are monotonic

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Solving the Consumer's Problem

- What is the intuition for this?

2. Tangency condition
   - If not, then the rate at which the **consumer** is willing to trade off good 1 and good 2 is different to the rate they can **trade them off in the market**
     - Example, say that the MRS is 0.5, but the price of each good is 1.
     - Can this be optimal? No
     - If the consumer consumed 1 less unit of good 1, then they could get 1 more unit of good 2
     - But they would only have to get 0.5 units to make them indifferent
     - Trading one unit of good 1 for one unit of good two is **feasible** and will make them **better off**

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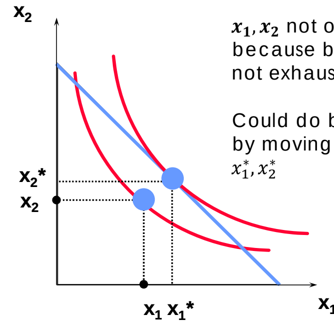
Non Optimal Choices



x_1, x_2 not optimal because budget not exhausted

Could do better by moving to x_1^*, x_2^*

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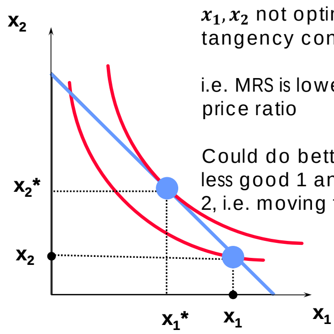
Non Optimal Choices



x_1, x_2 not optimal because tangency condition fails

i.e. MRS is lower that the price ratio

Could do better by buying less good 1 and more good 2, i.e. moving to x_1^*, x_2^*

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Solving the Consumer's Problem

- Do these two conditions always hold at the optimum?
- If preferences are (strictly or weakly) monotonic it **must** be optimal to spend all one's income
- What about the tangency condition?
  - Let's think about the case of perfect substitutes

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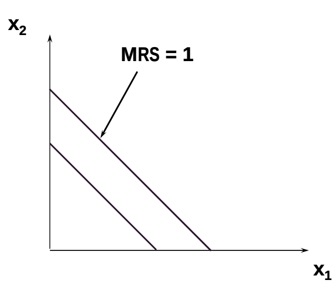
Examples of Corner Solutions -- the Perfect Substitutes Case



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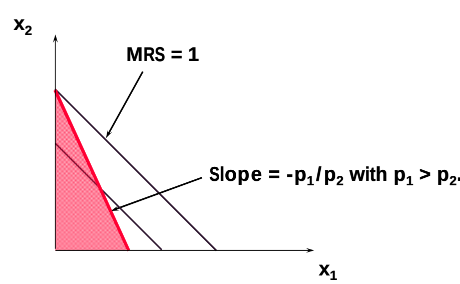

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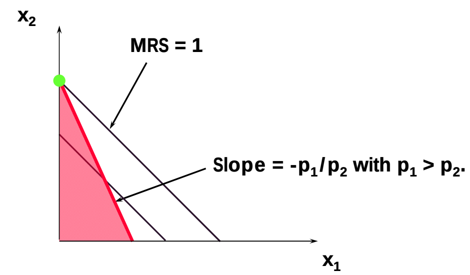

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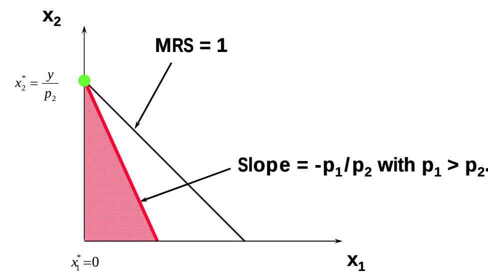

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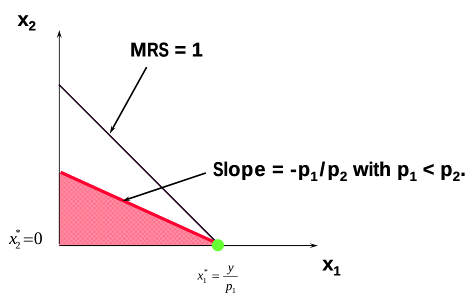

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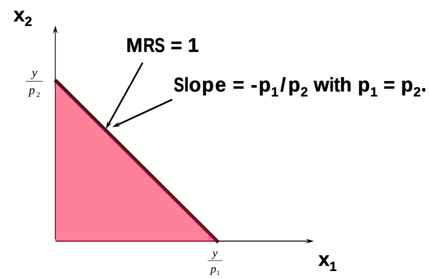

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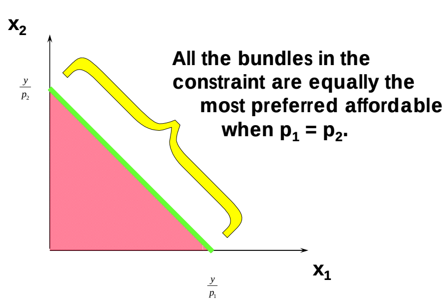

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Solving the Consumer's Problem

- In the case of perfect substitutes, the tangency condition does not (in general) hold
- What went wrong with our previous logic?
- Well, think about the case where MRS=1, but the price of good 1 is greater than the price of good 2
- The consumer would like to consume less of good 1 and more of good 2

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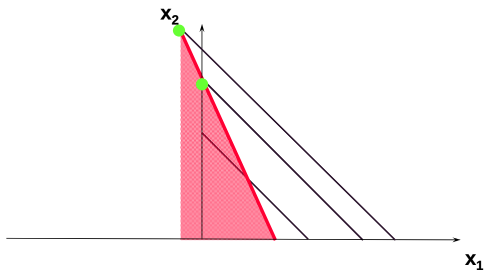
Examples of Corner Solutions -- the Perfect Substitutes Case



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Solving the Consumer's Problem

- However, once they are consuming zero units of good 1 they can no longer do so
- They cannot consume negative amounts of good 1
- They have hit the boundary of the commodity space
- This is called a **corner solution**

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Solving the Consumer's Problem

- Assuming that preferences are monotonic, there are two types of solution to the consumer's problem
  - **Interior** solutions
  - **Corner** solutions
- Three questions

1. Are perfect substitutes the only type of preferences that give us corner solutions?
2. Are tangency points always optimal?
3. Are interior solutions always tangency points?

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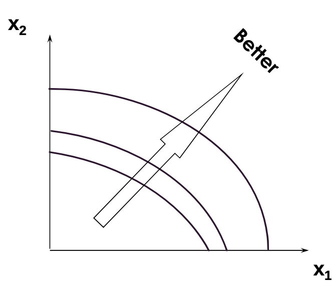

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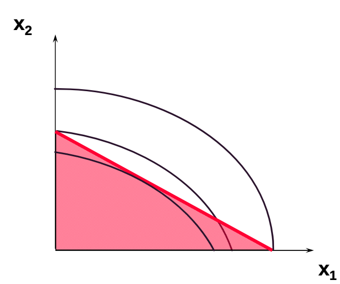

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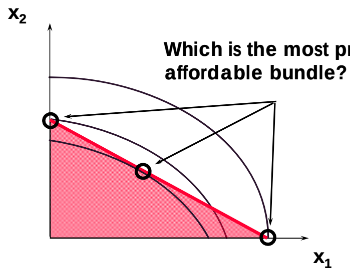

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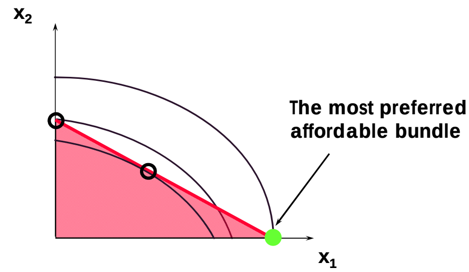

Solving the Consumer's Problem

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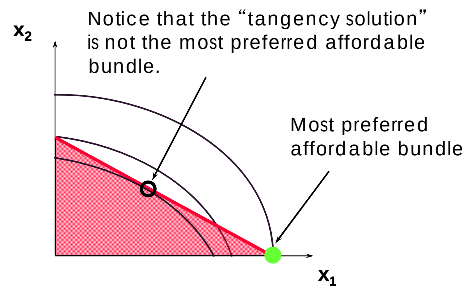

Solving the Consumer's Problem

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Perfect Complements Case

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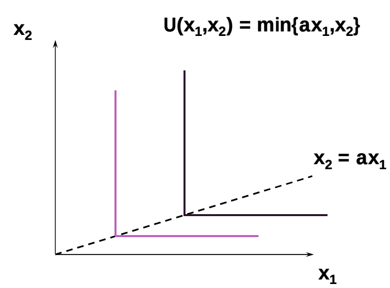

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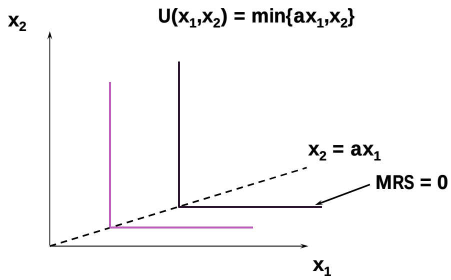
Perfect Complements Case



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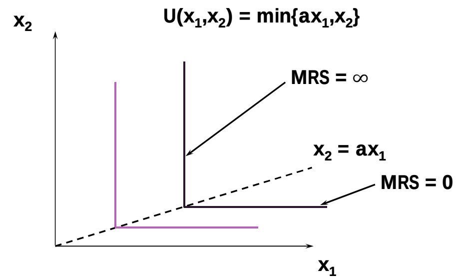

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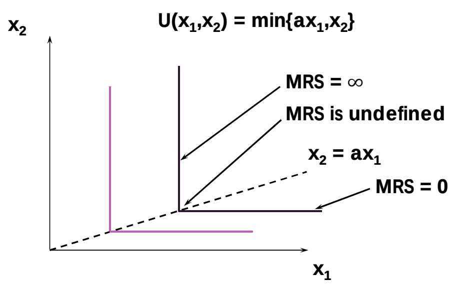

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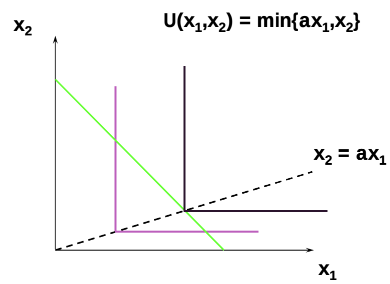

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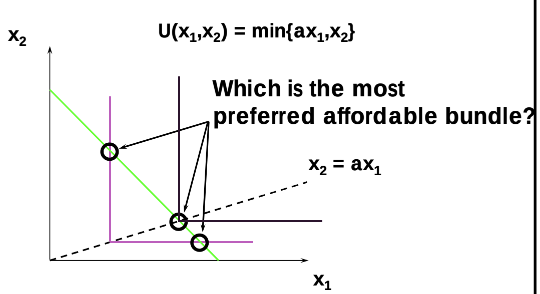

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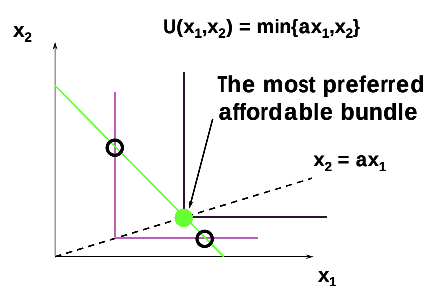

Perfect Complements Case

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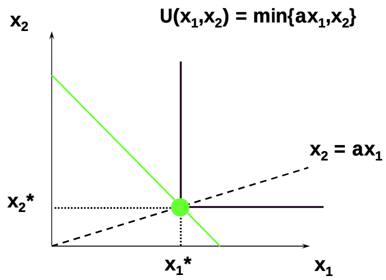

In general

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- Optimal bundles are such that they exhaust the budget (with monotone preferences)
- Indifference curves are also **often tangent** to the budget set at optimal bundles
  - i.e. ratio of prices is equal to the MRS
- But, tangency is **neither necessary nor sufficient**
  - Sometimes, optimum is **not** the point of tangency
    - Corner solutions
    - Kinks
  - Sometimes point of tangency is not an optimum
    - E.g. **concave** preferences

A Recipe for Solving the Consumer's Problem

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A Recipe for Solving the Consumer's Problem

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- Hopefully you now have some intuition about how to solve the consumer's problem
- Now we will get more formal about how to solve such problems
- I will provide you with a recipe for solving these problems
- If you follow these steps, you will find the right solution
- Over time, you will learn some short cuts
- However, be warned! I will try to fool you
  - Don't rely on the shortcuts without thinking!
- It is always a good idea to check your intuition as much as possible by drawing graphs of the type that we covered in the previous section

The Recipe

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1. Are preferences monotone?
   - If **yes**, then the optimal solution must lie on the budget line
   - If **no** you may have to worry about solutions away from the line
2. Assuming preferences are monotone, there are two possible types of solution
   - **Corner** solutions
   - **Interior** solutions
3. Calculate the utility at each possible corner solution
4. Find all possible interior solutions
   - Points of tangency
   - Kinks
5. Calculate utility and each possible interior solution
6. Compare utilities at **all** possible solutions
7. Select the best

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1: Check Whether Preferences are Monotone

- If preferences are monotone, life is easy
  - We know the optimal choice must lie on the budget line
- If not, then there are other possibilities
- How do we check whether preferences are monotone?
- Assuming we have a utility function, can check marginal utility

$$MU_i = \frac{\partial u}{\partial x_i} > 0$$

- [Question for you to think about. Is **weak** monotonicity okay?]
- If we find that preferences are monotone, we can go on to the next step
- If not what to do?
- Think!
  - Is one of the goods a bad? If so can you flip the problem around?
  - Is there a 'bliss point'? If so can they afford the bliss point?

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3: Calculate Utility for Corner Solutions

- As we have seen, it is possible that optimal solutions may occur at the corner of the commodity space
- A 'brute force' way to check these points is to calculate the utility at the corners
- If there are only two commodities, x_1 and x_2 then corner solutions are easy: either consume all of one good or all of the other
  - i.e. if income is M and prices are p_1 and p_2, calculate $u(\frac{M}{p_1}, 0)$ and $u(0, \frac{M}{p_2})$
- If there are three (or more) commodities, things get a bit more complex
  - Assume you are consuming 0 units of good 1, then figure out the optimal way of consuming goods 2 and 3
  - Repeat setting good 2 to 0
  - Repeat setting good 3 to 0

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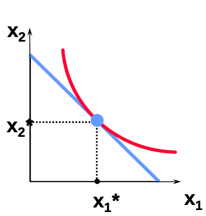
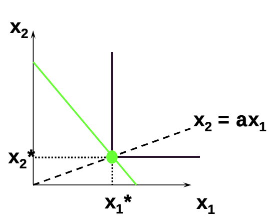
4: Find all Interior Solutions

- Apart from corner solutions, we potentially have interior solutions when the consumer buys some of all goods
- As we saw, these come in two types:
  - Tangency points (if the utility function is smooth)
  - At kinks (if the utility function is not differentiable)





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4: Find all Points of Tangency

- Remember: points of tangency occur when the slope of the budget line equals the slope of the indifference curve
  - Rate at which the consumer trades off the two goods is the same as the rate at which they can trade them off in the market
- How can we find points of tangency?
  - Slope of budget line is $\frac{-p_1}{p_2}$
  - Slope of indifference curve is $-MRS(x_1, x_2)$
- Furthermore, we know that

$$MRS(x_1, x_2) = -\frac{dx_2}{dx_1} = \frac{\partial u}{\partial x_1} \Big/ \frac{\partial u}{\partial x_2}$$

- Putting this together gives

$$\frac{\partial u}{\partial x_1} \Big/ \frac{\partial u}{\partial x_2} = \frac{p_1}{p_2}$$

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4: Find all Points of Tangency

- So points of tangency occur at

$$\frac{\partial u}{\partial x_1} \Big/ \frac{\partial u}{\partial x_2} = \frac{p_1}{p_2}$$

- Notice that prices p_1 and p_2 are parameters of the problem
  - They will appear in the solution
- But there are still two unknowns x_1 and x_2 and one equation
- How can we solve for both?
- Make use of another equation!
- Specifically the budget constraint
$$p_1 x_1 + p_1 x_1 = m$$
- We will see a worked example in a minute

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4: Find all the Kinks

- Sometimes, interior solutions may occur at kinks
  - E.g. perfect complements
- So we want to find all the kinks
- How can we spot a kink?
- Technically speaking kinks are points at which the indifference curve is not differentiable
- Easiest way to spot it is to graph the indifference curve
- Can also look out for points where the MRS doesn't seem to be well defined
  - Don't worry, we will only cover simple cases
- Once you have identified the kink, use the budget constraint to plug in to find values of x_1 and x_2

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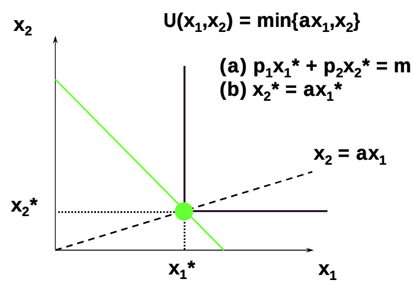
Perfect Complements Case



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6: Calculate the utility of all possible solutions

- Let's say that we found two points of tangency (x_1^*, x_2^*) and (x_1^{**}, x_2^{**}) (and no kinks)
- We can just plug these consumption bundles into the utility function
  - Calculate $u(x_1^*, x_2^*)$ and $u(x_1^{**}, x_2^{**})$
- We know that the optimal solution is **either** one of these two points **or** one of the corner solutions.
- Thus, all we need to do is to compare
  - $u(x_1^*, x_2^*)$
  - $u(x_1^{**}, x_2^{**})$
  - $u(\frac{M}{p_1}, 0)$
  - $u(0, \frac{M}{p_2})$
- And pick the best, and we are done!

A Worked Example

Your first solved consumer's problem!

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A Worked Example

- We will now apply the recipe to the following problem

1. **CHOOSE** $x_1 \geq 0, x_2 \geq 0$
2. **IN ORDER TO MAXIMIZE** $u(x_1, x_2) = x_1 x_2$
3. **SUBJECT TO** $3\,x_1 + x_2 = 4$

- You will want (and will have) lots more examples to get comfortable with this

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1: Check Whether Preferences are Monotone

- can check marginal utility

$$MU_1 = \frac{\partial u}{\partial x_1} = x_2 \geq 0$$

$$MU_2 = \frac{\partial u}{\partial x_2} = x_1 \geq 0$$

- Monotonicity is satisfied (strictly, apart from at the corners)

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The Recipe

1. Are preferences monotone?
   - If **yes**, then the optimal solution must lie on the budget line
   - If **no** you may have to worry about solutions away from the line
2. Assuming preferences are monotone, there are two possible types of solution
   - **Corner** solutions
   - **Interior** solutions
3. Calculate the utility at each possible corner solution
4. Find all possible interior solutions
   - Points of tangency
   - Kinks
5. Calculate utility and each possible interior solution
6. Compare utilities at **all** possible solutions
7. Select the best

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3: Calculate Utility for Corner Solutions

- Remember
  - $m = 4$
  - $p_1 = 3$
  - $p_2 = 1$
- If we spend all the money on good one we get

$$u\left(\frac{M}{p_1}, 0\right) = \frac{4}{3} \times 0 = 0$$

- If we spend all the money on good two we get

$$u\left(0, \frac{M}{p_2}\right) = 0 \times 4 = 0$$

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4: Find all Points of Tangency

- Points of tangency occur at

$$\frac{\partial u}{\partial x_1} \Big/ \frac{\partial u}{\partial x_2} = \frac{p_1}{p_2}$$

- Remember
  - $MU_1 = \frac{\partial u}{\partial x_1} = x_2 \geq 0$
  - $MU_2 = \frac{\partial u}{\partial x_2} = x_1 \geq 0$
- So

$$\frac{x_2}{x_1} = \frac{p_1}{p_2} \text{ or } x_2 = \frac{3}{1} x_1$$

- So we have solved for x_2 as a function of x_1

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4: Find all Points of Tangency

$$x_2 = 3x_1$$

- Now make use of the budget constraint

$$3\,x_1 + x_2 = 4$$

- So

$$3\,x_1 + 3\,x_1 = 4$$

- Or

$$x_1 = \frac{2}{3} \text{ and } x_2 = 2$$

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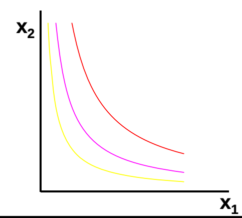
4: Find all the Kinks

- Do these indifference curves have kinks?
- No
- We can see this either by noticing that the $\frac{x_2}{x_1}$ is well behaved everywhere
- Or by graphing the indifference curves



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The Recipe

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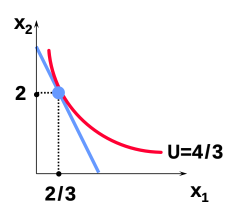
6: Calculate the utility of all possible solutions

- So we have three possibilities to check
  - $u\left(\frac{M}{p_1}, 0\right) = \frac{4}{3} \times 0 = 0$
  - $u\left(0, \frac{M}{p_2}\right) = 0 \times 4 = 0$
  - $u\left(\frac{2}{3}, 2\right) = \frac{2}{3} \times 2 = \frac{4}{3}$
- We have a winner!



Summary

Summary

- Today we have thought intuitively about how to solve the consumer's problem
- Introduced a recipe for solving such problems
- Worked through the various stages of solving this problem
- Next week: Some mathematical shortcuts!