Question 1 (Indifference Curves)

1. Assume that the consumer only gains utility from plants in plant pots. Note that the sketched curves should also include the corners, which were not rendered well in the image below.

**Montonicity:** Generally, it is easier to start showing the strict properties of the preferences since the strict version imply the respective weaker ones. Strict monotonicity tells us that adding any additional amount to any commodity of a consumption bundle will strictly increase its preference. Horizontal/vertical lines for an indifference curve are usually an indication that this does not hold as it implies that adding more of a particular would not take us away from the IC. Specifically, if we are at the consumption bundle (plants, pots) = (3, 1), adding any $\epsilon > 0$ amount of pots, would give us a consumer bundle of $(3 + \epsilon, 1)$ which is on the same IC as $(3, 1)$. Therefore $(3, 1) \sim (3 + \epsilon, 1)$, a violation of strict monotonicity. However, since these preferences are given by $\min\{\text{pots, } \frac{\text{plants}}{3}\}$, then adding both pots and plants would clearly take the consumer to a higher indifference curve (moving Northeast on the graph) and so preferences are weakly monotonic: if a bundle $x$ is composed of commodities strictly greater in quantity than those of another bundle $y$, then $x \succ y$.

**Convexity:** Strict convexity is a property in which for any two bundles $x$ and $y$ such that $x \sim y$, any mixture of the two $(\alpha x + (1 - \alpha)y; \alpha \in (0, 1))$ must be strictly better than $x$ and $y$. However, a convex combination from the same “flat” part of these indifference curves would always give us another point on the indifference curve, a violation of this property. Concretely, if we take the bundles $(3, 2)$ and $(3, 3)$, then any mixture (also called convex combination) of the two
will be of the form \((3, \beta)\), where \(\beta \in (2, 3)\).\(^1\) This is clearly on the same IC as \((3, 2)\) and \((3, 3)\), so \((3, 2) \sim (3, \beta) \sim (3, 3)\), a violation of strict convexity. On the other hand, no mixture of two consumer bundles result in a less-preferred bundle and the above example shows that there are mixtures that are equally preferred to the component bundles, so preferences are weakly convex.

In sum, preferences satisfy weak monotonicity but not strict monotonicity as well as weak convexity but not strict convexity.

2. Note that “cats” are plotted on the x-axis, the labels on the indifference curves denote the difference of cats and dogs and increase northwest. Preferences are weakly convex, not strict convex, and not monotonic\(^2\) The lack of monotonicity is due to the fact that dogs are a “bad” (that is, more dogs in a consumption bundle negatively impact bundle’s preference by the consumer).

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\(^1\)Make sure to understand why if this isn’t directly clear to you.

\(^2\)As in, they neither satisfy strict monotonicity and weak monotonicity.
3. The preferences for the consumer result in vertical indifference curves if we take the consumer’s bananas to be on the horizontal axis. In other words, the consumer does not care about the other’s number of bananas. These preferences satisfy weak (but not strict) monotonicity and weak (but not strict) convexity.

4. For a utilitarian consumer, the consumer’s banana and the friend’s banana are “perfect substitutes”. These preferences satisfy strict (and thus weak) monotonic-
ity and weak (but not strict) convexity.

5. These preferences include a *satiation point*: the consumer can do no better than receiving 6 chicken and 50 fries. These are not monotonic but satisfy strict (and thus weak) convexity.

6. These preferences are consistent with the notion of the consumer spending constant fractions of wealth on the commodities. Here, we only consider the situa-
tion where both goods are strictly positive. Preference are strictly (and therefore weakly) monotonic and strictly (and therefore weakly) convex.

Question 2 (Optional Consumption and Cobb Douglas)

1. The marginal utility of each utility function with respect to the goods is captured by the respective partial derivative.

- **Professor Hirst**
  
  \[
  MU_s = \frac{\partial u(x_s, x_c)}{\partial x_s} = \frac{1}{3}x_s^{-\frac{2}{3}}x_c^{\frac{2}{3}} = \frac{1}{3}(\frac{x_s}{x_c})^{\frac{2}{3}} \\
  MU_c = \frac{\partial u(x_s, x_c)}{\partial x_c} = \frac{2}{3}x_c^{-\frac{1}{3}}x_s^{\frac{1}{3}} = \frac{2}{3}(\frac{x_s}{x_c})^{\frac{1}{3}}
  \]

- **Professor Nilsson**
  
  \[
  MU_s = \frac{\partial v(x_s, x_c)}{\partial x_s} = 2(\frac{1}{x_s}) = \frac{2}{x_s} \\
  MU_c = \frac{\partial v(x_s, x_c)}{\partial x_c} = 4(\frac{1}{x_c}) = \frac{4}{x_c}
  \]

2. The Marginal Rate of Substitution of \( x_s \) for \( x_c \) is given by: \( MRS = \frac{MU_s}{MU_c} \).

   - **Professor Hirst**: 
     \[
     MRS = \frac{\frac{1}{3}(\frac{x_s}{x_c})^{\frac{2}{3}}}{\frac{2}{3}(\frac{x_s}{x_c})^{\frac{1}{3}}} = \frac{1}{2} \frac{x_c}{x_s}.
     \]
   
   - **Professor Nilsson**: 
     \[
     MRS = \frac{\frac{1}{x_c}}{\frac{1}{x_s}} = \frac{1}{2} \frac{x_s}{x_c}.
     \]

3. Letting spam be on the horizontal axis, the point of tangency occurs if the slope of the budget constraint \( -\frac{p_s}{p_c} \) must equal \(-MRS\):
• Professor Hirst:

\[
\frac{MU_s}{MU_c} = MRS = \frac{p_s}{p_c} \\
1 \frac{x_c}{x_s} = \frac{p_s}{p_c} \\
2 \frac{x_s}{x_c} = \frac{p_s}{p_c} \\
p_c x_c = 2p_s x_s
\]

The points of tangency of the indifference curve with the budget constraint must fulfill this condition above. Substituting, into the budget constraint, we get that:

\[
p_s x_s + p_c x_c = M \\
p_s x_s + (2p_s x_s) = M \\
x_s = \frac{M}{3p_s}.
\]

We can also not give explicit solve for \( x_c \):

\[
x_c = 2x_s \frac{p_s}{p_c} \\
x_c = 2(\frac{M}{3p_s}) \frac{p_s}{p_c} \\
x_c = \frac{2M}{3p_s}
\]

So the tangency point is \((x_s, x_c) = (\frac{M}{3p_s}, \frac{2M}{3p_c})\).

• Professor Nilssen:

\[
\frac{MU_s}{MU_c} = MRS = \frac{p_s}{p_c} \\
1 \frac{x_c}{x_s} = \frac{p_s}{p_c} \\
2 \frac{x_s}{x_c} = \frac{p_s}{p_c} \\
p_c x_c = 2p_s x_s
\]

The points of tangency of the indifference curve with the budget constraint must fulfill this condition above. Substituting, into the budget constraint, we
get that:

\[ p_s x_s + p_c x_c = M \]
\[ p_s x_s + (2p_s x_s) = M \]
\[ x_s = \frac{M}{3p_s} \]

We can also not give explicit solve for \( x_c \):

\[ x_c = \frac{2x_s p_s}{p_c} \]
\[ x_c = 2\left(\frac{M}{3p_s}\right) \frac{p_s}{p_c} \]
\[ x_c = \frac{2M}{3p_s} \]

So the tangency point is \((x_s, x_c) = (\frac{M}{3p_s}, \frac{2M}{3p_c})\). It is the same for both professors.

4. At this point, it should be clear that we are solving the solving identical optimization problems for each professor on the interior of the budget set. Recall that the consumption bundle can either be in the interior of the budget constraint\(^3\) (given by the tangency point) or a corner solution (where the bundle includes 0 of one commodity). We can check for the optimal consumption bundle by comparing the utilities under each professor’s utility function:

- **Professor Hirst:**
  - Interior: \((x_s, x_c) = (\frac{M}{3p_s}, \frac{2M}{3p_c})\). \[ u(\frac{M}{3p_s}, \frac{2M}{3p_c}) = \left(\frac{M}{3p_s}\right)^{1/3} (\frac{2M}{3p_c})^{2/3} > 0. \]
  - Corner when \(x_c = 0\): All of money \(M\) is spent on spam: \(x_s = \frac{M}{p_s}\). The utility is \(u(\frac{M}{p_s}, 0) = (\frac{M}{p_s})^{1/3}(0)^{2/3} = 0.\)
  - Corner when \(x_s = 0\): All of money is spent on chocolate: \(x_c = \frac{M}{p_c}\). The utility is \(u(0, \frac{M}{p_c}) = (0)^{1/3}(\frac{M}{p_c})^{2/3} = 0.\)

Since the tangency point gives the most utility, the optimal consumption bundle is \((x_s, x_c) = (\frac{M}{3p_s}, \frac{2M}{3p_c})\).

- **Professor Nilsson:**
  - Interior: \((x_s, x_c) = (\frac{M}{3p_s}, \frac{2M}{3p_c})\). \[ v(\frac{M}{3p_s}, \frac{2M}{3p_c}) = 2 \ln(\frac{M}{3p_s}) + 4 \ln(\frac{M}{3p_c}) > -\infty. \]
  - Corner when \(x_c = 0\): As above, then \(x_s = \frac{M}{p_s}\) and \(v(\frac{M}{p_s}, 0) = 2 \ln(\frac{M}{p_s}) + 4 \ln 0 = -\infty.\)

\(^3\)Different from the interior of the budget set.
Corner when \( x_s = 0 \): As above, then \( x_c = \frac{M}{p_c} \). Then \( v(0, \frac{M}{p_c}) = 2 \ln 0 + 4 \ln \left( \frac{M}{p_c} \right) = -\infty \).

Since the tangency point gives the most utility, the optimal consumption bundle is \((x_s, x_c) = \left( \frac{M}{3 p_s}, \frac{2 M}{3 p_c} \right)\).

There should be no disagreement between Hirst and Nilsson regarding Ermintrude’s preferences because, being monotonic transformations of each other, both \( u(x_s, x_c) \) and \( v(x_s, x_c) \) give the same optimal consumption bundles. This shows us that it is not the numerical “util” produced from bundles that is important in utility maximization; it is the optimal consumption selected when a particular utility function serves as a representation of underlying consumer preferences.

5. Knowing that \( v(x_s, x_c) = 6 \ln(u(x_s, x_c)) \) would make it apparent that \( u \) and \( v \) are related by a positive monotonic transformation, which according to Question 4 of Homework 1 mean that they represent the same preferences and therefore would yield the same optimal consumption bundle under the same budget constraint.

**Question 3 (Optimal Consumption and Quasi-Linear Preferences)**

1. Given some utility level (not a function) \( u \), the an indifference curve is given by \( x_g + (x_b)^{1/2} \). Rewriting as function banjos,

\[
x_g = u - (x_b)^{1/2}.
\]

2. The marginal rate of substitution \( MRS = \frac{MU_b}{MU_g} = \frac{\partial u}{\partial x} \). Note that \( \frac{\partial u}{\partial x} = \frac{1}{2} x_b^{-1/2} \) and \( \frac{\partial u}{\partial g} = 1 \), so

\[
MRS = \frac{1}{2 x_b^{1/2}}.
\]

3. Recall that from question 2, any point of tangency is given by when the slope of the indifference curve (MRS) is equal to the slope of the budget constraint (price
\[ MRS = \frac{p_b}{p_g} \]
\[ \frac{1}{2x_b^{1/2}} = \frac{p_b}{p_g} \]
\[ \frac{1}{2p_b} x_b^{1/2} = x_b^{1/2} \]
\[ x_b = \frac{1}{4} \left( \frac{p_g}{p_b} \right)^2. \]

4. A point of tangency between the indifference curve and budget constraint would yield an interior solution. We also need to check whether the tangency point is within the budget set. Substituting for the formula above,

\[ x_b = \frac{1}{4} \left( \frac{p_g}{p_b} \right)^2 = \frac{1}{4} \left( \frac{2}{1} \right)^2 = 1. \]

Using the budget constraint to solve for gun consumption at the tangency point gives us:

\[ p_b x_b + p_g x_g = M \]
\[ (1)(1) + 2x_g = 8 \]
\[ x_g = \frac{7}{2}. \]

Through its manipulation, we implicitly satisfied the budget constraint. We also clearly respect the non-negativity constraints on the goods. Finally, we check for corner solutions by comparing utilities:

- Interior: \( u(x_g, x_b) = u(3.5, 1) = 3.5 + 1^{1/2} = 4.5. \)
- Corner solution when \( x_g = 0 \): Spend all of \( M \) on banjos: \( x_b = \frac{M}{p_b} = 8. \) \( u(0, 8) = 0 + 8^{1/2} = 2.83. \)
- Corner solution when \( x_b = 0 \): Spend all of \( M \) on guns: \( x_g = \frac{M}{p_g} = 4. \) \( u(4, 0) = 4. \)

The tangency point is both feasible and gives the highest utility. The optimal consumption bundle is thus given by the interior solution:

\( (x_g, x_b) = (3.5, 1) \)

\(^4\)Writing the budget constraint as a function of the number of banjos: \( x_g = \frac{p_b}{p_g} x_b - \frac{M}{p_g} \)
5. At the point of tangency, the number of banjos would be:

\[ x_b = \frac{1}{4} \left( \frac{5}{1} \right)^2 = \frac{25}{4}. \]

However, when we solve for the number of guns under the given budget constraint:

\[
\begin{align*}
1 \left( \frac{25}{4} \right) + 5x_g &= 4 \\
5x_g &= 4 - \frac{25}{4} = -\frac{9}{4} \\
x_g &= -\frac{9}{20},
\end{align*}
\]

which violates the fact that consumption bundle must be non-negative in all commodities \((x_g = -\frac{9}{20} < 0)\). This is an indication that we have a corner solution for the optimal consumption bundle. In this case, Ezekiel consumes as little of guns as possible: \((x_g = 0)\) to get as close to the tangency point as he can.

In other words, comparing the interior solution with the possible corner solutions as before:

- Interior solution \((x_g, x_b) = \left( -\frac{9}{20}, \frac{25}{4} \right)\). \(u\left( -\frac{9}{20}, \frac{25}{4} \right) = -\frac{9}{20} + \frac{5}{2} = 2.05\).
- Corner when \(x_g = 0\): Spend all of \(M\) on banjos: \(x_b = \frac{M}{p_b} = 4\). \(u(0, 4) = 0 + 4^{1/2} = 2\).
- Corner \(x_b = 0\): Spend all of \(M\) on guns: \(x_g = \frac{M}{x_g} = \frac{4}{5}\). \(u(\frac{4}{5}, 0) = 0.8\)

We have that the interior solution gives the highest utility again, but since it is not feasible, we must select the highest utility out of the possible corner solutions, which yields the optimal consumption bundle of

\[(x_g, x_b) = 0\]

Graphically, we see that the interior solution is outside the feasible below. The blue and magenta lines respectively denote the indifference curves of feasible and infeasible utility, and the gray gives the budget set. Note the tangency only occurs with infeasible utility, but it is outside the budget set.