# 4 Lecture 4

## 4.1 Applications

We now will look at some of the applications of the convex analysis we have learned. First, we shall us a separation theorem to prove the second fundamental theorem of welfare economics

### 4.1.1 Second Fundamental Theorem of Welfare Economics

As you know from your micro class, the second fundamental welfare theorem states that any Pareto Optimal allocation of goods can be supported as a competitive equilibrium (under certain conditions). Here we are going to use a separation theorem to prove this result in the case of an $m$ person, $l$ commodity exchange economy. This economy is defined by a set $\{\omega^i, u_i\}_{i=1}^m$, where $\omega^i$ is an $l$ length vector of endowments for each person in the economy, and $u_i : \mathbb{R}^l_+ \to \mathbb{R}$ is a utility function for each person. We will make the following assumptions:

**Assumption 1** $\omega^i >> 0$ for all $i \in 1..m$

**Assumption 2** $u_i$ is continuous and strictly increasing for all $i$

**Assumption 3** for any $x, y \in \mathbb{R}^l_+$, $\alpha \in (0,1)$, $u(\alpha x + (1-\alpha)y) > \min(u(x), u(y))$ (this property is called strict quasi-concavity)

A (feasible) allocation is any vector $x = (x^1......x^m)$ where $x^i \in \mathbb{R}^l_+$ such that

$$\sum_{i=1}^m x^i = \sum_{i=1}^m \omega^i$$

An allocation is weakly Pareto optimal if

$$u_i(x^i) < u_i(y^i) \ \forall \ i \in 1..m$$

for no other feasible allocation $y$.

It is strongly pareto optimal if there is no allocation $y$ such that

$$\begin{aligned} u_i(x^i) &\leq u_i(y^i) \ \forall \ i \in 1,..,m \\ u_i(x^i) &< u_i(y^i) \text{ some } i \in 1,..,m \end{aligned}$$

Let $p$ be a price vector if $p \in \mathbb{R}^l_{++}$. Define the demand correspondence of agent $i$ as

$$d_i(p, \omega^i) = \arg\max \left\{ u_i(x^i) : x^i \in \mathbb{R}^l_+ \text{ and } px^i \leq p\omega^i \right\}$$

You should check that the theorem of the maximum, along with the strict q-concavity assumption ensures that these are all continuous functions.

A competitive equilibrium is a price vector $p$ and an allocation $x$ such that

$$x^i \in d_i(p, \omega^i) \ \forall \ i \in 1..m$$

As I am sure you all know, any competitive equilibrium is pareto efficient (the first fundamental welfare theorem). The key here is to show that any pareto optimal allocation can be supported as a competitive equilibrium (the second fundamental welfare theorem)

**Theorem 8 (The Second Fundamental Theorem of Welfare Economics)** *For any exchange economy in which A1-A3 hold, for any strongly Pareto optimal allocation $x_* \in \mathbb{R}^{l,m}_{++}$, there exists a price vector $p^*$ such that $x^i_* = d_i(p^*, x^i_*) \ \forall \ i \in 1, .., m$*

***Proof.*** *Let $x_*$ be a strongly pareto optimal allocation. Define*

$$S_i = \left\{ z \in \mathbb{R}^l_+ \text{ such that } u_i(z) > u_i(x^i_*) \right\}$$

*Note that $S_i$ is non-empty (by the fact that u is strictly increasing) and convex (due to strict q-concavity). Moreover, as u is continuous, $S_i$ is open. Let $S = S_1 + ... + S_m$, and note that this is also non-empty, convex and open. Finally, note that, as $x_*$ is strongly pareto optimal*

$$\sum_{i=1}^{m} x^i_* \notin S$$

*Otherwise, there would be an allocation y such that $y^i \in S^i \ \forall \ i$ and $\sum y^i = \sum x^i_* = \sum \omega^i$, but by construction, $u_i(y^i) > u_i(x^i)$ all i.*

*So we now have an open convex set S and a point $\sum_{i=1}^m x^i_* \notin S$. By corollary 3, we can separate S from $\sum_{i=1}^m x^i_*$, i.e. we can find a normal $p \in \mathbb{R}^l$ such that*

$$ps > p \sum_{i=1}^{m} x^i_* \ \forall \ s \in S$$

*The strictly increasing nature of* $u$ *also ensures that* $p \in \mathbb{R}_{++}$, *as, for every* $n \in 1, ..., l$

$$e^n + \sum_{i=1}^{m} x_*^i \in S$$

*so if* $p_n \leq 0$, *then we would have a contradiction.*

*Now all we have to do is show that* $x_i^* = d(p, x_*^i)\ \forall\ i$. *Suppose not, then there exists some* $i$ *and some bundle* $s^i \in \mathbb{R}_+^l$ *such that*

$$\begin{aligned} ps^i &\leq px_*^i \text{ and} \\ u_i(s^i) &> u_i(x_*^i) \end{aligned}$$

*By the continuity of* $u_i$, *we can therefore assume that* $ps^i < px_*^i$. *Define*

$$\theta = \frac{1}{m-1}(px_*^i - ps^i)$$

*and let*

$$s^j \in \arg\max \left\{ u_j(x^j) : x^j \in \mathbb{R}_+^l \text{ and } px^j \leq px_*^j + \theta \right\} \text{ all } j \neq i$$

*and note that, as* $u_j$ *is strictly increasing,* $u_j(s^j) > u_j(x_*^j)$ *and* $ps^j = px_*^j + \theta$. *But* $\sum^m s^j \in S$, *so*

$$\sum^m ps^j > \sum^m px_*^i$$

*But*

$$\begin{aligned} \sum^m ps^j &= p \sum^m s^j \\ &= ps^i + \sum^{m \neq i} ps^j \\ &= ps^i + \sum^{m \neq i} \left( px_*^j + \theta \right) \\ &= ps^i + \sum^{m \neq i} px_*^j + (m-1)\theta \\ &= \sum^m px_*^i \end{aligned}$$

*A contradiction* ■

### 4.1.2 Farkas' Lemma

We now move on to a second result: Farkas lemma. This is what we call a **theorem of the alternative,** which states that one and only one of two systems of equations can have a solution. This has all sorts of uses, but we will use it later to prove the Kuhn Tucker theorem

**Theorem 9 (Farkas' Lemma)** *Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, and consider the following two systems of equations*

1. *For $x \in \mathbb{R}^n$*

$$\begin{aligned} Ax &= b \\ x &\geq 0 \end{aligned}$$

2. *For $\lambda \in \mathbb{R}^m$*

$$\begin{aligned} \lambda^T A &\leq 0 \\ \lambda^T b &> 0 \end{aligned}$$

*Exactly one of these two systems will have a solution*

The reason that this is a useful result is that it means that, to prove 1 has a solution, it is enough to show that 2 does not.

We are going to prove this result in a sequence of claims. First we will prove that at most one of these two systems has a solution

**Lemma 8** *System 1 and system 2 cannot both have a solution*

***Proof.*** *By contradiction: Let $x$ be a solution to 1 and $\lambda$ be a solution to 2, then*

$$0 < \lambda^T b = \lambda^T A x \leq 0$$

■

In order to prove that one of these two systems must have a solution, we are going to do the following: Let

$$\begin{aligned} K &= A(\mathbb{R}^n_+) \\ &= \left\{y \in \mathbb{R}^m | y = Ax,\, x \in \mathbb{R}^n_+\right\} \end{aligned}$$

Clearly, system 1 only has a solution if and only if $b \in K$. We are going to show that system 2 has a solution if and only if $b \notin K$.

In order to do this, we are going to show that $K$ is a convex, closed cone, and in order to do *that*, we are going to have to define what a cone is:

**Definition 6** *A set $S \subset \mathbb{R}^n$ is a cone if, $\forall\ s \in S$, $\alpha > 0$, $\alpha s \in S$*

Why this is called a cone should be obvious if you think about the shape of such a set in $\mathbb{R}^2$.

**Lemma 9** $K = A(\mathbb{R}^n_+)$ *is a closed, convex cone.*

***Proof.*** *Note, that, for any $y_1$, $y_2 \in K$, there exits an $x^1$, $x^2 \in \mathbb{R}^n_+$ such that*

$$y_1 = \begin{matrix} a_{11}x_1^1 + ... + a_{1n}x_n^1 \\ \vdots \\ a_{m1}x_1^1 + ... + a_{mn}x_n^1 \end{matrix}$$

$$y_2 = \begin{matrix} a_{11}x_1^2 + ... + a_{1n}x_n^2 \\ \vdots \\ a_{m1}x_1^2 + ... + a_{mn}x_n^2 \end{matrix}$$

*From this, the fact that $K$ is convex and a cone should be obvious. For closedness, we will consider two cases. First, assume that all the columns in $A$ are linearly independent. Let $\left\{y^k\right\}$ be a sequence is $K$ such that $y^k \to y$. We need to show $y \in K$.*

*Let $x_l$ be the sequence such that $y_l = Ax_l$. We need to show that $x_l$ is bounded. If not, there exists a subsequence such that $||x_l|| \to \infty$. Let*

$$z_l = \frac{x_l}{||x_l||}$$

*$z_l$ is bounded and so has a convergent subsequence, so assume that $z_l \to z$, and note that $||z|| = 1$. But*

$$Ax_l = ||x_l||Az_l \to y \text{ and } ||x_l|| \to \infty$$

*so $Az_l \to 0$ implies that $Az = 0$, contradicting the linear independence of the columns of $A$. Thus, $x_l$ is bounded and, as $\mathbb{R}^n_+$ is closed, $x_l \to x \in \mathbb{R}^n_+$ and $y = Ax \in K$*

*Now consider the case where the columns of $A$ are not linearly independent. For any $y \in K$, there exists a subset of columns $\Theta \subset \{1,..,n\}$ such that*

$$\begin{aligned} y &= \sum_{j \in \Theta} x_j A_j \\ & \text{and } \{A_j\}_{j \in \Theta} \text{ is LI} \end{aligned}$$

*To see this, let $J_+ = \{j | x_j > 0\}$ If $\{A_{.,j} | j \in J_+\}$ is not LI, then there exists a $Z \in \mathbb{R}^n$ such that $z_j = 0 \ \forall \ j \notin J_+$, $Az = 0$*

*Consider $x - \alpha z$ where $\alpha = \min \left\{ \frac{x_j}{y_j} | z_j > 0 \right\}$ and $A(x - \alpha z) = y$, and $x - \alpha z = 0$ in one row. Running this enough times kills off all the LI columns of $A$.*

*Thus, let*

$$\sum = \{\Theta \subset \{1,..,n\} \, | A_\Theta \text{ is LI}\}$$

*and let*

$$K_\Theta = \{Ax | x \geq 0 \text{ and } x_j = 0 \ \forall \ j \notin \Theta\}$$

*Then*

$$K = \cup_{\Theta \in \sum} K_\Theta$$

*which is a finite union of closed sets and therefore closed* ■

We are now in a position to prove the theorem

**Proof (Farkas' Lemma).** We are going to show that system two has a solution if and only if $b \notin K$. As $K$ is closed and convex, then $K = cl(co(K))$, and so is the intersection of all closed half spaces that contain it. Hence, $b \notin K$ if and only if $\exists \ \lambda \neq 0$ such that $< \lambda, b >= \alpha$ and $\alpha >< \lambda, k >$ $\forall \ k \in K$.

Since $K$ is a cone, $\alpha > 0$. If $\alpha < 0$, then pick some $k$ such that $0 > \alpha >< \lambda, k >$. There exists a $\beta > 0$ such that $< \lambda, \beta k >> \alpha$ (just make $\beta$ close enough to zero). And as $K$ is a cone, $\beta k \in K$. Note also that we can construct a sequence $\{k_i\}_{i=1}^{\infty}$ such that $k_i \in K$ and $k_i \to 0$. Thus, as $K$ is closed, $0 \in K$, and therefore $\alpha \neq 0$. This in turn imples that

$$< \lambda, k > \leq 0 \forall k \in K$$

If not, then we could find some $k$ such that $< \lambda, k >> 0$, but as $< \lambda, \beta k >= \beta < \lambda, k >$, we can find an element in $K$ that would violate $< \lambda, \beta k >< \alpha$

Substituting in gives us

$$< \lambda, Ax > \leq 0 \ \forall \ x \in \mathbb{R}_+^n$$

and so $\lambda^T A \leq 0$ ■