Mathematics For Economists

Mark Dean

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Question 1 (30 pts) This question is about the continuity of correspondences

1. We can think of a function as a special case of a correspondence. If a function f mapping one metric space to another is upper-hemi continuous at a point x, is it continuous (i.e. for every $\varepsilon > 0$, $\exists\ \delta > 0$ such that $d(x, x') < \delta \Rightarrow \rho(f(x), f(x')) < \varepsilon$)? What about if it is lower hemi continuous? if the function is continuous, is it upper hemi continuous? Lower hemi-continuous? Prove or find a counter example in each case

2. In class, we claimed that a correspondence $\Gamma : X \rightrightarrows Y$ is lower hemicontinuous at $x \in X$ if and only if, for any sequence $x_m \to x$ in X, and any $y \in \Gamma(x)$, there exists a sequence $y_m \to y$ such that $y_m \in \Gamma(x_m)\ \forall\ x_m$, but we never proved it. Do so now.

3. Define a budget correspondence in $\mathbb{R}^n$ as $B : \mathbb{R}^{n+1}_{++} \rightrightarrows \mathbb{R}^n_+$ as

$$B(p, I) = \left\{x \in \mathbb{R}^n_+ | px \leq I\right\}$$

   (a) Is the budget correspondence compact valued? Prove or disprove

   (b) We want to show that the budget set is upper hemi-continuous (i) consider a sequence of budget sets defined by $p_m \to p$ and $I_m \to I$. Let $p_i^* = \inf(p_i^m | m \in \mathbb{N})$ and $I^* = \sup(I^m | m \in \mathbb{N})$. show that $p_i^* > 0\ \forall\ i$ and $I^* < \infty$. (ii) Now consider the budget set $B(p^*, I^*)$. Show that any sequence $x_m \in B(p^m, I_m)$ must have a subsequence that converges to a point in $B(p^*, I^*)$. (iii) use this to conclude that the budget correspondence is upper hemi-continuous.

   (c) Is the budget set lower hemi-continuous? Prove or disprove

4. Consider the correspondence $\Gamma : [0,1] \rightrightarrows [0,1]$ defined by

$$\begin{aligned} \Gamma(x) &= [0,1] \cap \mathbb{Q} \text{ if } x \in [0,1] \backslash \mathbb{Q} \\ &= [0,1] \backslash \mathbb{Q} \text{ if } x \in [0,1] \cap \mathbb{Q} \end{aligned}$$

Show that this correspondence is not continuous, but is lower hemi-continuous. Is it upper hemi-continuous at any $x \in \mathbb{Q}$? what about at any $x \in [0,1] \backslash \mathbb{Q}$. In each case prove either way.

Question 2 (20 points) Define the binary operations $+_1$ and $+_2$ on $\mathbb{R}^2$ by $x +_1 y = (x_1 + y_1, x_2 + y_2)$ and $x +_2 y = (x_1 + y_1, 0)$. define the operation $\cdot$ on $(\lambda, x) \in \mathbb{R} \times \mathbb{R}^2$ as $\lambda \cdot x = (\lambda x_1, \lambda x_2)$.Is $(\mathbb{R}^2, +_i, \cdot)$ a linear space for $i \in 1, 2$?. What if $\lambda \cdot x = (\lambda x_1, x_2)$? What if $\lambda \cdot x = (\lambda x_1, 0)$?

Question 3 (50 points) Consider the following problem: a firm has to choose a location to for a shop on block island. As suggested by its name, block island is square in shape, with each side of the square having length 10. There is currently one shop on block island located exactly in its centre. The firm wants to maximize their distance from the existing shop. However, the shop has to be within a certain distance from one of two supply depots The first supply depot s_1 is located at map reference $[5, 7]$ (where $[0, 0]$ is the bottom left hand corner of the island), while the second one, s_2 is located at $[7, 5]$. The shop has to be either located less than or equal to distance d_1 from s_1 or d_2 from s_2. In all cases, use the standard Euclidian metric for distances.

1. Write this problem as a maximization problem. Draw a graph to describe the problem
2. Is this problem guaranteed to have a solution from any value of d_1, d_2? Prove or find a counterexample
3. Assume (if you need to) that the problem has a solutionLet $f(d_1, d_2)$ be the maximal obtainable distance of the shop from the existing shop, and $L(d_1, d_2)$ be the set of optimal locations. Is f continuous? Is it strictly monotonic? For what values of d_1 and d_2 is L continuous?
4. Write this problem in the form of a KKT constrained maximization problem. Note that the type of constrained maximization problem we have considered does not allow for constraints of the 'either x must hold or y must hold' variety, so you may have to set up two optimization problems, depending on the vale of d_1 and d_2

5. For some value of d_1 and d_2, there will be points that are not regular. Find the values of d_1 and d_2 for which there are non-regular points, and show where these points are
6. Solve for the KKT first order conditions of the problems set up in part 4
7. Are the first order conditions necessary and sufficient to find the solution?
8. Fix a value of $d_1 = \bar{d}_1$. Write down an expression for the function $f(\bar{d}_1, d_2)$