Final 2011 - Suggested Solutions[1]

Question 1

See question 2 Homework 4

Question 2.-

Define the binary operations $+_1$ and $+_2$ on $\mathbb{R}^2$ by $x +_1 y = (x_1 + y_1, x_2 + y_2)$ and $x +_2 y = (x_1 + y_1, 0)$. Define the operation $\cdot$ on $(\lambda, x) \in \mathbb{R} \times \mathbb{R}^2$ as $\lambda \cdot x = (\lambda x_1, \lambda x_2)$. Is $(\mathbb{R}^2, +_i, \cdot)$ a linear space for $i \in 1, 2$? What if $\lambda \cdot x = (\lambda x_1, x_2)$? What if $\lambda \cdot x = (\lambda x_1, 0)$

Definition 1 (Linear space) *Let V be a non empty set. The list $(C, +, \cdot)$ is a linear space if $+$ is a binary operation on V and $\cdot$ is a mapping that assigns each $(\lambda, v) \in \mathbb{R} \times V$ and element $\lambda \cdot v$ of V (which we denote λv) such that for any $\alpha, \lambda \in \mathbb{R}$ and $v, w, z \in V$ the following properties hold:*

- *Additive properties*
  - *(associativity)* $(x + y) + z = x + (y + z)$
  - *(existence of a zero element) There exists an element* $0 \in X$ *such that* $0+x = x = x+0$ *for all* $x \in X$
  - *(existence of inverse elements) For each* $x \in X$*, there exits an element* $-x \in X$ *such that* $x + -x = 0 = -x + x$
  - *(commutative)* $x + y = y + x$ *for all* $x, y \in X$
- *Scalar multiplication properties*
  - *(associativity)* $\alpha(\lambda x) = (\alpha\lambda)x$
  - *(distributivity)* $(\alpha + \lambda)x = \alpha x + \lambda x$ *and* $\lambda(x + y) = \lambda x + \lambda y$
  - *(The unit rule)* $1x = x$

Let $\lambda \cdot_1 x = (\lambda x_1, \lambda x_2)$, $\lambda \cdot_2 x = (\lambda x_1, x_2)$ and $\lambda \cdot_3 x = (\lambda x_1, 0)$.

$+_1$ and λ_1

These are the typical operations and satisfy all the conditions in the definition above. (You need to check them!)

[1] If you find any typo please email me: Maria_Jose_Boccardi@Brown.edu

$+_2$ and λ_1

It is not a linear space, for example it doesn't satisfy the conditions of existence of a zero element. That is for any $x = (x_1, x_2)$ such that $x_2 \neq 0$, we have that $y + x = (x_1 + y_1, 0) \neq x$ for all $y \in \mathbb{R}^2$

$+_1$ and λ_2

It satisfies all the additive properties as in the case for $+_1$ and λ_1, and you can check (you should) that is satisfies all the scalar multiplication properties.

$+_2$ and λ_2

It is not a linear space, idem as $+_2$ and λ_1

$+_1$ and λ_3

It is not a linear space, given that it does not satisfies the existence of a unit element, that is for any $x = (x_1, x_2)$ such that $x_2 \neq 0$, $1x = (x_1, 0) \neq x$

$+_2$ and λ_3

It is not a linear space, idem as $+_2$ and λ_1

Question 3

Part 1

Write this problem as a maximization problem. Draw a graph to describe the problem.

The maximization problem can be written as

$$\max_{x \in A} \sqrt{(x_1 - 5)^2 + (x_2 - 5)^2} \tag{1}$$

where

$$A(d_1, d_2) \equiv \left(\overline{B}\left((5,7), d_1\right) \cup \overline{B}\left((7,5), d_2\right)\right) \cap block \tag{2}$$

where *block* refers to the island, that is

$$block \equiv \{y \in \mathbb{R}^2_+ : x_1, x_2 \leq 10\}$$

This problem is represent in figure 1

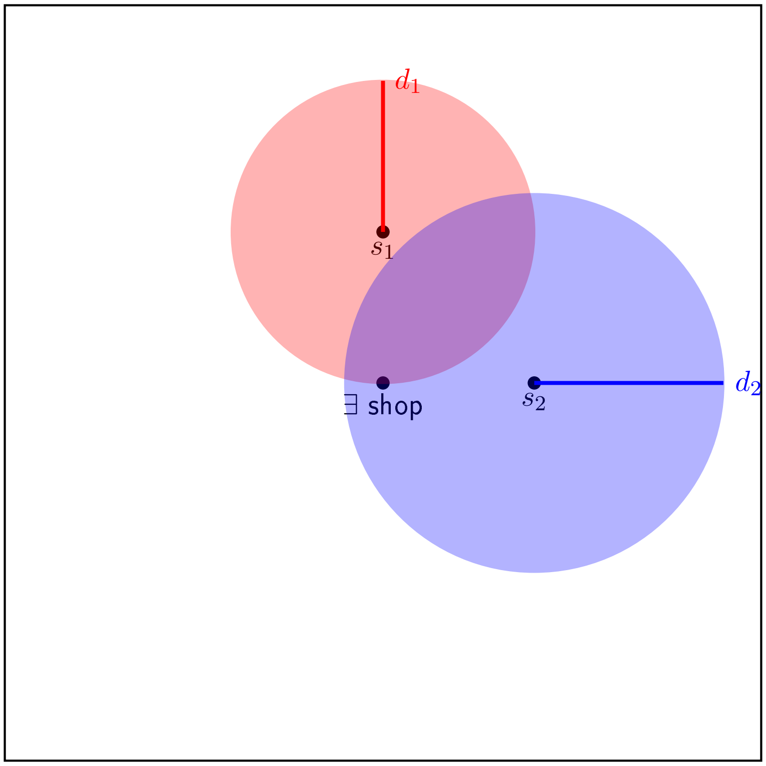

Figure 1: Description of the problem

Part 2.-

Is the problem guaranteed to have a solution for any value d_1, d_2? Prove or find a counterexample. For any value of d_1, d_2, $\overline{B}\left((5,7), d_1\right)$ and $\overline{B}\left((7,5), d_2\right)$ are closed sets, then $\overline{B}\left((5,7), d_1\right) \cup \overline{B}\left((7,5), d_2\right)$ is a closed set, and since $block$ is a closed set, A defined as in 2 is a closed set, and moreover, since $block$ is bounded it is also a bounded set. The objective function is continuous, therefore by Weierstrass we have guaranteed the existence of a solution for this problem.

Part 3.-

Assume (if you need to) that the problem has a solution. Let $f(d_1, d_2)$ be the maximal obtainable distance of the shop from the existing shop, and let $L(d_1, d_2)$ be the set of optimal locations. Is f continuous? Is it strictly monotonic? For what values d_1 and d_2 is L continuous?

Theorem 2 (Theorem of the Maximum) *Let*

- X *and* Y *be metric spaces*
- $\Gamma : X \to Y$ *be compact valued and continuous*
- $f : X \times Y \to \mathbb{R}$ *be continuous*

Now define $y^* : X \to Y$ as the set of maximizers of f given parameters x

$$y^*(x) = arg \max_{y \in \Gamma(x)} f(x, y)$$

and define $f^* : X \to Y$ as the maximized value of f for f given parameters x

$$f^*(x) = \max_{y \in \Gamma(x)} f(x, y)$$

Then

1. *y^* is upper hemicontinuous and compact valued*
2. *f^* is continuous*

Let $X = \mathbb{R}^2_+$, $Y = A(d_1, d_2)$, Γ the correspondence defined by 2, and finally, $f(x_1, x_2, d_1, d_2) = d(x, (5, 5))$ assuming the euclidean metric and therefore f is continuous. We are in the conditions of the theorem of the maximum, you just need to show that the correspondence is continuous (with respect to d_1, d_2). Then we have that $f(d_1, d_2)$ is a continuous function.

f is not necessarily strictly monotonic. For example, if d_1, d_2 are such that $A = block$ then further changes in d_1, d_2 won't impact the value of the function at the optimum. For example, consider $d_1 = d_2 = 10$, and $d_1 = d_2 = 10 + \varepsilon$ does not affect the value of the function at the optimum.

For free, also from the theorem of the maximum we have that L is upperhemicontinuous. For continuity we need to prove that L is lowerhemicontinuous. We are going to have that for some values of d_1, d_2 the correspondence L is not lower hemicontinuous. Consider for example the sequence $(d_1^m, d_2^m) = (3 + \frac{1}{m}, 3) \to (3, 3) \in X$ and consider $(10, 5) \in L(3, 3)$ (show why), then there is not a sequence of y_m such that converges to $(5, 10)$ because for any m the optimum will be given by $x \in \overline{B}((5, 7), d_1^m)$. See figure 2. In particular if d_1, d_2 are such that they always keep the same order then the correspondence is continuous.

Part 4.-

Write this problem in the form of a KKT constrained maximization problem. If $d_1 < d_2$ then the problem is to

$$\max \sqrt{(x_1 - 5)^2 + (x_2 - 5)^2} \tag{3}$$

subject to

$$\sqrt{(x_1 - 7)^2 + (x_2 - 5)^2} \leq d_2 \tag{4}$$

and

$$x_1, x_2 \geq 0 \text{ and } x_1, x_2 \geq 10 \tag{5}$$

If $d_2 < d_1$ then the problem is to

$$\max \sqrt{(x_1 - 5)^2 + (x_2 - 5)^2} \tag{6}$$

subject to

$$\sqrt{(x_1 - 5)^2 + (x_2 - 7)^2} \leq d_1 \tag{7}$$

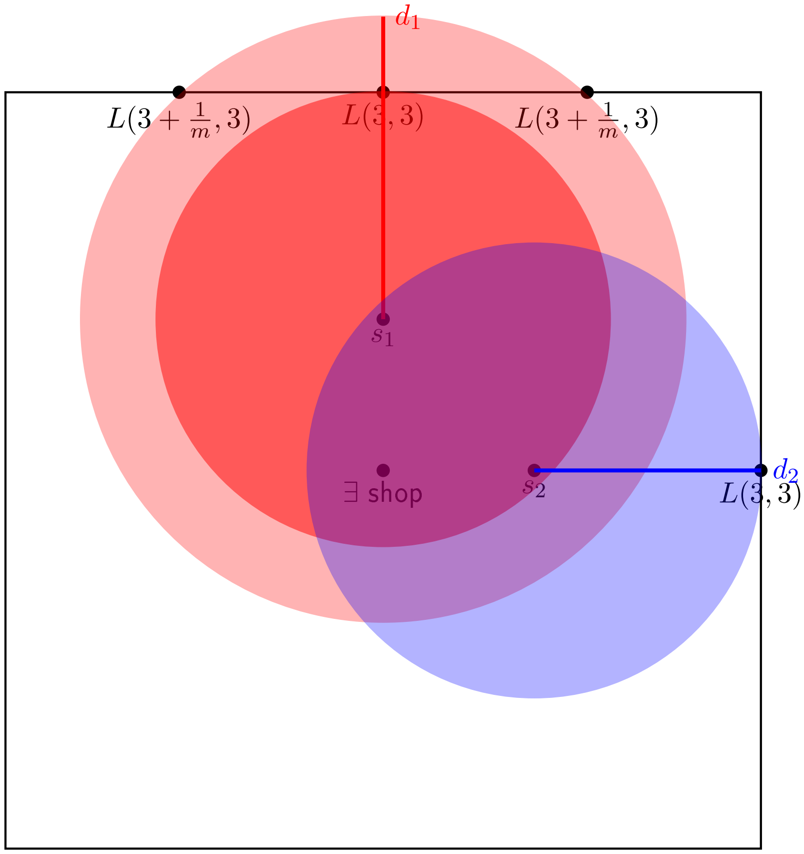

Figure 2: $L(d_1, d_2)$ is not lowerhemicontinous

and

$$x_1, x_2 \geq 0 \text{ and } x_1, x_2 \geq 10 \tag{8}$$

Finally if $d_1 = d_2$ then the solution would be the union of the solutions to the two problems described above.

Part 5.-

For some values of d_1 and d_2, there will be points that are not regular. Find the values of d_1 and d_2 for which there are non regular points and show where these points are.

If d_1, d_2 are strictly positive, if $d_1 = 3$ then $x = (5, 10)$ is non regular, as well as if $d_2 = 3$ then $x = (10, 5)$ is non regular. (why?)

Part 6.-

Solve for the KKT first order conditions of the problem set up in part 4 Consider the first problem defined above, then the first order conditions are. For simplicity, consider the maximization of the squared of the objective functions describe above

$$\begin{aligned}
2(x_1-5)+\mu_1\frac{1}{2}\frac{2(x_1-7)}{\sqrt{((x_1-7)^2+(x_2-5)^2)}}-\mu_2+\mu_4 &= 0\\
2(x_2-5)+\mu_1\frac{1}{2}\frac{2(x_2-5)}{\sqrt{((x_1-7)^2+(x_2-5)^2)}}-\mu_3+\mu_5 &= 0\\
\mu_i\leq 0 & \qquad \forall i\\
\mu_1\left(\sqrt{(x_1-7)^2+(x_2-5)^2}-d_2\right) &= 0\\
\mu_2(-x_1) &= 0\\
\mu_3(-x_2) &= 0\\
\mu_4(x_1-10) &= 0\\
\mu_5(x_2-10) &= 0
\end{aligned}$$

Idem for the other problem.

Part 7.-

In order to have sufficiency we need to ensure that the second order conditions are satisfied. For example $x_1 = x_2 = 5$ satisfies the conditions from part 6 with all $\mu_i = 0$ but that's clearly a minimum.

Part 8.-

Fix a value of $d_1 = \overline{d}_1$. Write down an expression for the function $f(\overline{d}_1, d_2)$.

Consider first the case where $d_2 < \overline{d}_1$. If $\overline{d}_1 \leq 3$ then the optimal will we given by $(5, 7+\overline{d}_1)$ and then $f(\overline{d}_1, d_2) = 2+\overline{d_1}$. If $\sqrt{34} \geq \overline{d}_1 > 3$ then the optimal will be given by $bdry\left(\overline{B}(s_1, \overline{d}_1)\right) \cap bdry(block)$, if $\sqrt{34 \overline{d}_1} > \sqrt{74}$ then optimal will be given by $(0, 10)$ and $(10, 10)$ and $f(\overline{d}_1, d_2) = \sqrt{50}$ while if $\overline{d}_1 \geq \sqrt{74}$ the optimal is given by all the extreme points of the block, and $f(\overline{d}_1, d_2) = \sqrt{50}$

Similarly for the case of $d_2 \geq \overline{d}_1$.