Final 2011 - Suggested Solutions

Question 1

See question 2 Homework 4

Question 2.

Define the binary operations $+_1$ and $+_2$ on $\mathbb{R}^2$ by $x+_1 y = (x_1 + y_1, x_2 + y_2)$ and $x+_2 y = (x_1 + y_1, 0)$. Define the operation $\cdot$ on $(\lambda, x) \in \mathbb{R} \times \mathbb{R}^2$ as $\lambda \cdot x = (\lambda x_1, \lambda x_2)$. Is $(\mathbb{R}^2, +_i, \cdot)$ a linear space for $i \in 1, 2$? What if $\lambda \cdot x = (\lambda x_1, x_2)$? What if $\lambda \cdot x = (\lambda x_1, 0)$?

Definition 1 (Linear space) Let $V$ be a non empty set. The list $(C, +, \cdot)$ is a linear space if $+$ is a binary operation on $V$ and $\cdot$ is a mapping that assigns each $(\lambda, v) \in \mathbb{R} \times V$ and element $\lambda \cdot v$ of $V$ (which we denote $\lambda v$) such that for any $\alpha, \lambda \in \mathbb{R}$ and $v, w, z \in V$ the following properties hold:

- **Additive properties**
  - (associativity) $(x + y) + z = x + (y + z)$
  - (existence of a zero element) There exists an element $0 \in X$ such that $0 + x = x = x + 0$ for all $x \in X$
  - (existence of inverse elements) For each $x \in X$, there exits an element $-x \in X$ such that $x + (-x) = 0 = -x + x$
  - (commutative) $x + y = y + x$ for all $x, y \in X$

- **Scalar multiplication properties**
  - (associativity) $\alpha (\lambda x) = (\alpha \lambda)x$
  - (distributivity) $(\alpha + \lambda)x = \alpha x + \lambda x$ and $\lambda(x + y) = \lambda x + \lambda y$
  - (The unit rule) $1x = x$

Let $\lambda_1 x = (\lambda x_1, \lambda x_2)$, $\lambda_2 x = (\lambda x_1, x_2)$ and $\lambda_3 x = (\lambda x_1, 0)$.

$+_1$ and $\lambda_1$

These are the typical operations and satisfy all the conditions in the definition above. (You need to check them!)

\[1\]

If you find any typo please email me: Maria_Jose_Boccardi@Brown.edu
+2 and $\lambda_1$

It is not a linear space, for example it doesn’t satisfy the conditions of existence of a zero element. That is for any $x = (x_1, x_2)$ such that $x_2 \neq 0$, we have that $y + x = (x_1 + y_1, 0) \neq x$ for all $y \in \mathbb{R}^2$.

+1 and $\lambda_2$

It satisfies all the additive properties as in the case for +1 and $\lambda_1$, and you can check (you should) that it satisfies all the scalar multiplication properties.

+2 and $\lambda_2$

It is not a linear space, idem as +2 and $\lambda_1$.

+1 and $\lambda_3$

It is not a linear space, given that it does not satisfies the existence of a unit element, that is for any $x = (x_1, x_2)$ such that $x_2 \neq 0$, $1x = (x_1, 0) \neq x$.

+2 and $\lambda_3$

It is not a linear space, idem as +2 and $\lambda_1$.

**Question 3**

**Part 1**

Write this problem as a maximization problem. Draw a graph to describe the problem.

The maximization problem can be written as

$$\max_{x \in A} \sqrt{(x_1 - 5)^2 + (x_2 - 5)^2}$$

where

$$A(d_1, d_2) \equiv (\overline{B}((5, 7), d_1) \cup \overline{B}((7, 5), d_2)) \cap \text{block}$$

where block refers to the island, that is

$$\text{block} \equiv \{ y \in \mathbb{R}_+^2 : x_1, x_2 \leq 10 \}$$

This problem is represent in figure 1.
Part 2.-

Is the problem guaranteed to have a solution for any value $d_1, d_2$? Prove or find a counterexample. For any value of $d_1, d_2$, $B((5, 7), d_1)$ and $B((7, 5), d_2)$ are closed sets, then $B((5, 7), d_1) \cup B((7, 5), d_2)$ is a closed set, and since block is a closed set, $A$ defined as in 2 is a closed set, and moreover, since block is bounded it is also a bounded set. The objective function is continuous, therefore by Weierstrass we have guaranteed the existence of a solution for this problem.

Part 3.-

Assume (if you need to) that the problem has a solution. Let $f(d_1, d_2)$ be the maximal obtainable distance of the shop from the existing shop, and let $L(d_1, d_2)$ be the set of optimal locations. Is $f$ continuous? Is it strictly monotonic? For what values $d_1$ and $d_2$ is $L$ continuous?

**Theorem 2 (Theorem of the Maximum)** Let

- $X$ and $Y$ be metric spaces
- $\Gamma : X \to Y$ be compact valued and continuous
- $f : X \times Y \to \mathbb{R}$ be continuous
Now define \( y^* : X \rightarrow Y \) as the set of maximizers of \( f \) given parameters \( x \)

\[
y^*(x) = \arg \max_{y \in \Gamma(x)} f(x, y)
\]

and define \( f^* : X \rightarrow Y \) as the maximized value of \( f \) for \( f \) given parameters \( x \)

\[
f^*(x) = \max_{y \in \Gamma(x)} f(x, y)
\]

Then

1. \( y^* \) is upper hemicontinuous and compact valued
2. \( f^* \) is continuous

Let \( X = \mathbb{R}^2_+ \), \( Y = A(d_1, d_2) \), \( \Gamma \) the correspondence defined by \( \Box \) and finally, \( f(x_1, x_2, d_1, d_2) = d(x, (5, 5)) \) assuming the euclidean metric and therefore \( f \) is continuous. We are in the conditions of the theorem of the maximum, you just need to show that the correspondence is continuous (with respect to \( d_1, d_2 \)). Then we have that \( f(d_1, d_2) \) is a continuous function.

\( f \) is not necessarily strictly monotonic. For example, if \( d_1, d_2 \) are such that \( A = \text{block} \) then further changes in \( d_1, d_2 \) won’t impact the value of the function at the optimum. For example, consider \( d_1 = d_2 = 10 \), and \( d_1 = d_2 = 10 + \varepsilon \) does not affect the value of the function at the optimum.

For free, also from the theorem of the maximum we have that \( L \) is upperhemicontinuous. For continuity we need to prove that \( L \) is lowerhemicontinuous. We are going to have that for some values of \( d_1, d_2 \) the correspondence \( L \) is not lower hemicontinuous. Consider for example the sequence \( (d_1^m, d_2^m) = (3 + \frac{1}{m}, 3) \rightarrow (3, 3) \in X \) and consider \( (10, 5) \in L(3, 3) \) (show why), then there is not a sequence of \( y_m \) such that converges to \( (5, 10) \) because for any \( m \) the optimum will be given by \( x \in \overline{B}((5, 7), d_1^m) \). See figure \( \Box \). In particular if \( d_1, d_2 \) are such that they always keep the same order then the correspondence is continuous.

Part 4.-

Write this problem in the form of a KKT constrained maximization problem. If \( d_1 < d_2 \) then the problem is to

\[
\max \sqrt{(x_1 - 5)^2 + (x_2 - 5)^2}
\]

subject to

\[
\sqrt{(x_1 - 7)^2 + (x_2 - 5)^2} \leq d_2
\]

and

\[
x_1, x_2 \geq 0 \text{ and } x_1, x_2 \geq 10
\]

If \( d_2 < d_1 \) then the problem is to

\[
\max \sqrt{(x_1 - 5)^2 + (x_2 - 5)^2}
\]

subject to

\[
\sqrt{(x_1 - 5)^2 + (x_2 - 7)^2} \leq d_1
\]
and

\[ x_1, x_2 \geq 0 \text{ and } x_1, x_2 \geq 10 \] (8)

Finally if \( d_1 = d_2 \) then the solution would be the union of the solutions to the two problems described above.

**Part 5.**

For some values of \( d_1 \) and \( d_2 \), there will be points that are not regular. Find the values of \( d_1 \) and \( d_2 \) for which there are non regular points and show where these points are.

If \( d_1, d_2 \) are strictly positive, if \( d_1 = 3 \) then \( x = (5, 10) \) is non regular, as well as if \( d_2 = 3 \) then \( x = (10, 5) \) is non regular. (why?)

**Part 6.**

Solve for the KKT first order conditions of the problem set up in part 4. Consider the first problem defined above, then the first order conditions are. For simplicity, consider the maximization of the squared of the objective functions describe above.
\[ 2(x_1 - 5) + \mu_1 \frac{1}{2} \frac{2(x_1 - 7)}{\sqrt{(x_1 - 7)^2 + (x_2 - 5)^2}} - \mu_2 + \mu_4 = 0 \]

\[ 2(x_2 - 5) + \mu_1 \frac{1}{2} \frac{2(x_2 - 5)}{\sqrt{(x_1 - 7)^2 + (x_2 - 5)^2}} - \mu_3 + \mu_5 = 0 \]

\[ \mu_i \leq 0 \quad \forall i \]

\[ \mu_1 \left( \sqrt{(x_1 - 7)^2 + (x_2 - 5)^2} - d_2 \right) = 0 \]

\[ \mu_2(-x_1) = 0 \]

\[ \mu_3(-x_2) = 0 \]

\[ \mu_4(x_1 - 10) = 0 \]

\[ \mu_5(x_2 - 10) = 0 \]

Idem for the other problem.

**Part 7.-**

In order to have sufficiency we need to ensure that the second order conditions are satisfied. For example \( x_1 = x_2 = 5 \) satisfies the conditions from part 6 with all \( \mu_i = 0 \) but that’s clearly a minimum.

**Part 8.-**

Fix a value of \( d_1 = \overline{d}_1 \). Write down an expression for the function \( f(\overline{d}_1, d_2) \).

Consider first the case where \( d_2 < \overline{d}_1 \). If \( \overline{d}_1 \leq 3 \) then the optimal will be given by \( (5, 7 + \overline{d}_1) \) and then \( f(\overline{d}_1, d_2) = 2 + \overline{d}_1 \). If \( \sqrt{34} \geq \overline{d}_1 > 3 \) then the optimal will be given by \( bdry(B(s_1, \overline{d}_1)) \) \( \cap bdry(block) \), if \( \sqrt{34d_1} > \sqrt{74} \) then optimal will be given by \( (0, 10) \) and \( (10, 10) \) and \( f(\overline{d}_1, d_2) = \sqrt{50} \) while if \( \overline{d}_1 \geq \sqrt{74} \) the optimal is given by all the extreme points of the block, and \( f(\overline{d}_1, d_2) = \sqrt{50} \).

Similarly for the case of \( d_2 \geq \overline{d}_1 \).