Mathematics For Economists

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Homework 1

Due Tuesday 16th September

Question 1 Let $(X, \times, +)$ be a field.

1. Show that $0 \times x = 0$ for all $x \in X$
2. Show that if there exists an element $0^{-1} \in X$ such that $0^{-1} \times 0 = 1$, then $x = 0 \; \forall \; x \in X$

Question 2 We claimed in class that for any $a, b \in \mathbb{R}$, such that $a < b$, there exists a $q \in \mathbb{Q}$ such that $a < q < b$ (this is sometimes described as $\mathbb{Q}$ being order-dense in $\mathbb{R}$). Prove this statement. Also prove that there must exist an $r \in \mathbb{R}/\mathbb{Q}$ between a and b (i.e. the irrational numbers are also order-dense in $\mathbb{R}$)

Question 3 In class we introduced the concept of a binary relation. Consider the binary relation $\succeq$ on $X = \mathbb{R} \times [1, 2]$ defined by

$$\begin{aligned} \{a, i\} &\succeq \{b, j\} \text{ if and only if} \\ \text{(i) } a &> b \text{ or} \\ \text{(ii) } a &= b \text{ and } i \geq j \end{aligned}$$

These are sometimes called lexicographic preferences, because they work like a dictionary. When comparing any two alternatives, one first checks the first number, and uses that to determine preferences. If the first numbers are identical, then the second number is used to determine preferences. Show these preferences are transitive, reflexive and complete (i.e.$\{a, i\} \succeq \{b, j\}$ or $\{b, j\} \succeq \{a, i\}$ for all $\{a, i\}, \{b, j\}$ in X)

A utility function $u : X \to \mathbb{R}$ is said to represent a binary relation on X if $u(x) \geq u(y)$ if and only if $x \succeq y$ for all $x, y \in X$. Usually, if a binary relation is complete, transitive and reflexive, then it can be represented by a utility function. However, if X is uncountable, that may not be true. In fact, the lexicographic preferences cannot be represented by a utility function, as you are going to prove. You are going to do this by contradiction. Imagine that u does represent the lexicographic preferences described above. Note that, for any $a \in \mathbb{R}$, it must be the case that $u(a, 2) > u(a, 1)$. Now use Theorem 3 from the real numbers lecture notes to derive a contradiction (we did not formally prove theorem 3, so you should do that as well. You can assume that every non-degenerate interval on the real line contains a rational (i.e. part 2 of proposition 3).

Question 4 We define a complete preference relation $\succeq$ on a space X as a complete, tranistive and reflexive binary relation. We also define the binary relation $\succ$ as the asymmetric part of $\succeq$ - i.e. $x \succ y$ if $x \succeq y$ but not $y \succeq x$. We consider a preference relation to be continuous on a metric space if, for any x and y such that $x \succ y$, there exists a radius r such that $\bar{x} \succ \bar{y}$ for any $\bar{x} \in B(x, r)$ and $\bar{y} \in B(y, r)$.

1. Consider the preferences defined in question 3. The first question is: are these preferences defined on a metric space? Well, they are defined on the cartesian product of two spaces $\mathbb{R} \times [1, 2]$, both of which have metrics, so we might expect the answer to be yes. In fact, for the cartesian product for any number of metric spaces $X_1 \times X_2 \times ... \times X_n$ with associated metrics $d_1, d_2....$, we can define the product metric for some index $p > 0$ as

$$d_p\left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}\right) = \left(\sum_{i=1}^{n} d_i(x_i, y_i)^p\right)^{\frac{1}{p}}$$

   Show that the product metric is, in fact, a metric

2. Apply the metric $d(x, y) = |x - y|$ to both $\mathbb{R}$ and $[1, 2]$, and apply the d_1 product metric to $\mathbb{R} \times [1, 2]$. Are the preferences in question 1 continuous?

3. Now apply the discrete metric to both $\mathbb{R}$ and $[1, 2]$ and apply the d_1 product metric to $\mathbb{R} \times [1, 2]$. Are the preferences in question 1 continuous?