Mathematics For Economists

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Homework 2

Due Tuesday, 23rd September

Question 1 Prove the following properties of closures and interior points

1. $cl(A) = \{x \in M | x \text{ is a closure point of } A\}$
2. $int(A) = \{x \in M | \exists\ r > 0 \text{ such that } B(x, r) \subset A\}$
3. If $A \subset B$ then $cl(A) \subseteq cl(B)$. However, if $cl(A) \subseteq cl(B)$ then it is not necessarily the case that $A \subseteq B$
4. For A,$B \subset M$, then $cl(A \cup B) = cl(A) \cup cl(B)$ and $cl(A \cap B) \subseteq cl(A) \cap cl(B)$ but it is not necessarily the case that $cl(A \cap B) = cl(A) \cap cl(B)$
5. What is the closure of $\mathbb{Q}$?

Question 2 Some properties of continuous functions.

1. Let f and g be continuous functions. Is $f + g$ a continuous function? What about $f \times g$? $f.g$? (i.e. the function $h(x)$ defined as $h(x) = f(g(x))$
2. Let $u : X \to \mathbb{R}$ be a continuous utility function. Show that the upper and lower contour sets (i.e. the sets $\{x : u(x) \geq y)$ and $\{x : u(x) \leq y\}$ are closed for any $y \in \mathbb{R}$. Is it true that the sets $\{x : u(x) > \alpha)$ and $\{x : u(x) < \alpha\}$ are necessarily open?
3. Show that, if f is a continuous function on an interval $[a, b]$ then it is uniformly continuous. Give an example of a function on a closed but unbounded interval that is continuous but not uniformly continuous.

4. Show that the function $f(x) = x^{\frac{1}{2}}$ on $[0,1]$ is uniformly continuous but not Lipschitz continuous

Question 3 Let P be some finite set of strictly positive price vectors for N goods and I be a finite set of income levels. Let $\mathcal{D}$ be the set of demand functions - i.e. functions $D : P \times I \rightarrow \mathbb{R}^N$ such that

$$\sum_i D_i(p,i)p_i \leq i$$

Define a metric on $\mathcal{D}$

$$d_\infty(D, D') = \sup_{p.i \in \{P,I\}} d(D(p,i), D'(p,i))$$

where d is some metric on $\mathbb{R}^N$

1. Show that d_∞ is a metric
2. Let π be a probability distribution on $P \times I$ and $u : \mathbb{R}^N \rightarrow \mathbb{R}$ be a utility function. The expected utility of a demand function is given by

$$U(D) = \sum_{p.i \in \{P,I\}} \pi(p,i)u(D(p,i))$$

   Show that, in general, U is not continuous.
3. Provide conditions on u such that U is continuous. Provide an example by which you can change whether U is continuous for a particular u by changing the metric d