Mathematics For Economists

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Homework 3

Due October 7th

Question 1 Some lim sup and lim inf questions

1. Find the lim sup and lim inf of the following sequences. Prove your claims using one of the equivalent definitions we used in class
   (a) $\{x_i\}_{i=1}^{\infty}$ such that $x_i = -1$ if i is odd, and $\frac{1}{i}$ if i is even
   (b) $\{x_i\}_{i=1}^{\infty}$ such that $x_i = -1$ if i is odd, and i if i is even
   (c) $x_i = \left(-\frac{1}{i}\right)^i$
2. Let $\{x_i\}$ be a bounded sequence of real numbers, and define $y_i = \frac{1}{i}\sum_{j=1}^{i} x_j$ (i.e. the average of the first i elements of x).Show that $\limsup y_i \leq \limsup x_i$

Question 2 In the lecture notes, we stated the following: If a metric space M is separable then there exists a countable collection of open sets $\mathcal{O}$ such that, for any open subset U of M

$$U = \cup \{O \in \mathcal{O} | O \subseteq U\}$$

Prove this statement (hint: think of the set of open balls that have their center at a point in the countable dense subset of M and have rational radius)

Question 3 Here, we will prove a generalization of the intermediate value theorem.

1. Let A be a subset of $\mathbb{R}$. Show that A is connected if and only if it is an interval
2. Let X, Y be two metric spaces and $f : X \to Y$. Show that, if f is continuous and X is connected, then $f(X)$ is a connected subset of Y

3. Use these two facts to show the following: Let X be a connected metric space, and $f : X \to \mathbb{R}$ be a continuous function. if, for some $a \in \mathbb{R}$ and $x, y \in X$ $f(x) \leq a \leq f(y)$, there exists a $z \in X$ such that $f(z) = a$
4. Use this result to show that, for any continuous function $f : [a, b] \to [a, b]$, there exists some c such that $f(c) = c$ (i.e. the function has a fixed point)

Question 4 Here are some things for you to show regarding compact sets

1. Show that every compact metric space is complete
2. Show that continous functions map compact sets to compact sets
3. Does the finite union of compact sets have to be compact? What about an arbitrary union of compact sets