Mathematics For Economists

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Homework 5

Due Tuesday 28th Oct

Question 1 Let A be an $n \times m$ matrix. Prove the following:

1. A system of equations $Ax = c$ has a solution if and only if $c \in \text{col}(A)$
2. A system of equations only has a solution for any c if $\text{col}(A)$ spans $\mathbb{R}^n$, and so $\dim(\text{col}(A)) = n = rank(A)$
3. If this is the case, then $n = rank(A) \leq m$
4. The space of solutions to any system of equations is an affine manifold
5. The dimention of that set of solutions is equal to $m - rank(A)$

Question 2 Show that the orthogonal complement of the null space of a matrix A is the row space of that matrix.

Question 3 This question refers to the application of linear algebra to asset pricing we covered in class. In that section we defined a set of l underlying commodities, and a set of n securities, where a security y_i is an l-length vector of commodities. We said that q was an n length vector of security prices, and a portfolio θ was an n length vector denoting the quantities owned of each security. We said that a set of commodity prices allowed an arbitrage opportunity if there existed a portfolio θ such that

$$\begin{aligned} \sum \theta_i y_i &= 0 \\ \theta.q &> 0 \end{aligned}$$

Prove the following theorem:

Theorem 1 *The security price vector q satisfies the non-arbitrage condition if and only if there is a vector $p \in \mathbb{R}^l$ such that*

$$q_j = \sum_{i \in 1}^{l} p_i y_j^i$$

for every $j \in 1, ..n$

Hint. Construct an $l \times m$ matrix of securites, and think of the no arbitrage condition saying something about (i) the nullspace of that matrix and (ii) the orthogonality of θ and q. How does the nullspace of a matrix relate to its row space?

What is the interpretation of the vector p?