Mathematics For Economists

Mark Dean

Homework 7

Due Tuesday 18th Nov

Question 1 Let (a, b) be an open interval of $\mathbb{R}$. Let $\mathcal{C}^1$ be the set of continuously differentiable functions on (a, b), and $\mathcal{C}$ be the set of continuous functions on (a, b)

1. Show that $\mathcal{C}^1$ is a strict subset of $\mathcal{C}$
2. Show that the derivative operator is an element of $\mathcal{L}(\mathcal{C}^1, \mathcal{C})$.
3. Show that the following function is differentiable everywhere on $(-1, 1)$, but its derivative is not continuous

$$\begin{aligned} f(x) &= x^2 \sin \frac{1}{x} \text{ for } x \neq 0 \\ &= 0 \text{ for } x = 0 \end{aligned}$$

(hint, you may need to use the squeeze theorem (look it up))

Question 2 Let $C \subset \mathbb{R}^n$ be convex. A function $f : C \to \mathbb{R}$ is convex if, for every $x, y \in C$, $\lambda \in (0, 1)$

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

It is strictly convex if the inequality is strict

1. The epigraph of a function $g : A \to \mathbb{R}$, where $A \subset \mathbb{R}^n$ is defined as

$$epi(g) = \{(x, y\}|x \in A \text{ and } y \geq g(x)\}$$

Show that a function f is convex if and only if its epigraph is a convex set

2. A convex function is necessarily continuous on an open set C. Show that this is not true for an arbitrary C

3. Let $C \subset \mathbb{R}$. Show that, if f is twice differentiable, then it is convex if and only if its second derivative is non-negative (hint, show that the first derivative is a non-decreasing function). Is it the case that it is strictly convex if and only if its second derivative is strictly positive?

Question 3 Let $A \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^m \setminus \{0\}$, $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$. Show that one and only one of the following systems has a solution:

$$(I)\ Ax = c$$
$$(II)\ A^T y = 0, \qquad \langle c, y \rangle = 1$$

(Hint: In order to use Farkas lemma, think of rewriting x as a $2n$ length vector, were the the first n elements are $x_i^+ = \max\{x_i, 0\}$ and the second n elements are $x_i^- = -\min\{x_i, 0\}$)