Mathematics For Economists

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Homework 8

Due Thursday Dec 4th

Question 1 Consider the following equality constraints in $\mathbb{R}^3$

$$\begin{aligned} h_1 &: x_1 - x_2 = 0 \\ h_2 &: x_3 - x_1^2 = 0 \\ h_3 &: x_3 - x_2^2 - \alpha = 0 \\ h_4 &: x_3 - \beta x_1 = 0 \end{aligned}$$

1. Consider the set $X = \left\{x \in \mathbb{R}^3 | h(x) = 0\right\}$. What does this set look like if $\alpha > 0$?. Assume $\alpha = 0$. What does the set look like if $\beta = 1$?
2. Let $\alpha = 0$, and consider the set $X' = \left\{x \in \mathbb{R}^3 | h_i(x) = 0, \, i = 1..3\right\}$ (i.e. ignore constraint 4) Is the constraint set regular at any point?
3. Characterize the set of tangent curves that pass through the point $x^* = (1, 1, 1)$. Use this characterization to derive the tangent plane at this point
4. Now calculate the kernal at the same point. How does this relate to the to the tangent plane?
5. Now consider the following set of constraints in $\mathbb{R}^2$

$$\begin{aligned} h_1 &: x_1^2 - x_2 = 0 \\ h_2 &: x_1^4 - x_2 = 0 \end{aligned}$$

   What does this constraint set look like?

6. What is the only parametric curve that goes through this set? Therefore, what is the tangent plane of this set at $(0,0)$?

7. Is this constraint set regular at $(0,0)$? What is the kernal at this point?

8. What does this tell you about the relationship of the regularity condition to the equivalence of the kernal and the tangent plane? Can you give some intuition as to what regularity is buying us?

Question 2 $y_1, ..., y_m$ are distinct points in $\mathbb{R}^n$. The aim is to find the smallest closed ball, $\overline{B(x,\rho)}$, that contains these points. This can be formulated as

$$\begin{aligned} & \min\left(\rho^2\right) \\ & s.t. \\ \left|\left|x - y^j\right|\right|^2 - \rho^2 &\leq 0 \ \forall \ j \in 1..m \\ -\rho &\leq 0 \end{aligned}$$

So you are choosing a location for the centre of the ball (x) and a radius of the ball ρ

- Let $\binom{x}{\rho} \in \mathbb{R}^{n+1}$ be a vector x and a value ρ. Characterize $X \subset \mathbb{R}^{n+1}$ as the feasible set for the optimization problem. Use Weierstrass theorem to show that this problem has a solution (hint - you may have to compactify)

- Using the Kuhn Tucker Necessary conditions, show that for a critical point (x_0, ρ_0) it must be the case that there m numbers $\mu_1^0, ...\mu_m^0$ such that

  - $\Sigma_j \mu_j^0 = 1$;
  - $x_0 = \Sigma_j \mu_j^0 y^j$;
  - $\rho_0^2 + ||x_0||^2 = \Sigma_j \mu_j \left|\left|y^j\right|\right|^2$.

  (Hint: For the last equation play with complimentary slackness)

- Let $f(x,\rho)$ be the value of the objective function at x, ρ, and $g_j(x,\rho)$ be the value of each constraint $j \in \{1,..m\}$ define

$$F((x,\rho),\mu) = f(x,\rho) + \sum_{j=1}^{m} \mu_j g_j(x,\rho)$$

Use the above result to show that we can restrict this function to

$$F((x,\rho),\mu) = ||x||^2 - 2 < x, \sum_{j=1}^{m} \mu_j y_j >$$

to verify the second order conditions

- Does this problem have a unique solution?
- Solve the special case $y^j = e^j$, for $j = 1, 2, 3$.

Question 3 Let f be a continuously differentiable function on an interval I in $\mathbb{R}$. Show that f is concave if and only if

$$f(y) - f(x) \leq f'(x)(y - x)$$

for all $x, y \in I$

Show that this implies that, if f is a continuously differentiable function on a convex subset U of $\mathbb{R}^n$, then f is concave on U if and only if, for all $x, y \in U$

$$f(y) - f(x) \leq \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f(x)(y_i - x_i)$$

(for this second part, you may want to prove that f is concave if and only if the function $g_{x,y}(t) = f(tx + (1-t)y)$ is concave for every $x, y \in U$)

Question 4 Let $f : \mathbb{R}^n \to \mathbb{R}$ be a concave and continuously differentiable function, and $g_j : \mathbb{R}^n \to \mathbb{R}$, $j = 1, ..., m$, be convex and continuously differentiable functions. Consider the optimization problem (P):

$$\max(f(x))$$

$$s.t.$$

$$g(x) \leq 0$$

Let $X = \mathbb{R}^n$, $U = \mathbb{R}^n_-$ and $l : X \times U \to R$ be the (Lagrangian) function

$$l(x,\mu) = f(x) + \Sigma_j \mu_j g_j(x)$$

A saddle point for l is a vector $(x^*, \mu^*) \in X \times U$ that satisfies

$$l(x,\mu^*) \leq l(x^*,\mu^*) \leq l(x^*,\mu)$$

for all $(x,\mu) \in X \times U$.

1. Show that if (x^*, μ^*) is a saddle point for l then x^* is an optimal solution for (P).

2. Conversely, assume that (P) satisfies the Slater constraint qualification condition (SCQ) ; there exists $\overline{x}$ such that $g(\overline{x}) < 0$. Show that if x^* solves (P) then there exists a μ^* such that (x^*, μ^*) is a saddle point for l (hint, use the answer to question 3, and think about the KKT conditions)