# 2 Lecture 2

## 2.1 Span, Basis and Dimensions

Related to the concept of a linear combination is that of the **span.** The span of a collection of objects is the set of all linear combinations of those objects

**Definition 5** *Let $(V, +, \bullet)$ be a linear space. The span of $S \subset V$ is defined as follows:*

$$span(S) = \left\{ \sum_{i=1}^{k} c_i x_i \middle| \begin{array}{c} k \in \mathbb{N} \\ c_1, ..., c_k \in \mathbb{R} \\ x_1, ..., x_k \in S \end{array} \right\}$$

It should be obvious that the span of any set $S \subset V$ forms a linear subspace of $V$. What might be less obvious is that it is the *smallest* linear subspace that contains $S$, but this should be obvious from the fact that any subspace is closed under addition and scalar multiplication.

In $\mathbb{R}^2$, the span of any single vector is the line that goes through the origin and that vector.[2] The span of any two vectors in $\mathbb{R}^2$ is generally equal to $\mathbb{R}^2$ itself. This is only not true if the two vectors lie on the same line - i.e. they are linearly dependent, in which case the span is still just a line. This is a demonstration of an important property: adding linearly dependent elements to a set does not increase its span. In fact, more generally all objects that are linear combinations of that set will already be in its span.

**Proposition 1** *Let $S, T \subset V$ where $V$ is a linear space. If every element of $T$ is a linear combination of elements of $S$ then*

1. *$T \in span(S)$*
2. *$span(S \cup T) = span(S)$*

***Proof.*** *Part one follows obviously from the definition (check). To prove part two, we need to show both that $span(S \cup T) \subseteq span(S)$ and $span(S \cup T) \supseteq span(S)$. The second part is obvious (check).*

[2] This brings up the question: what is the span of the vector $(0, 0)$? The answer is that, by convention, we set the span of the zero element to equal only that element in any linear space.

*To see the first part, let $x \in span(S \cup T)$. This means that*

$$x = \left\{ \sum_{i=1}^{k} c_i x_i | \begin{array}{c} k \in n \\ c_1, ..., c_k \in \mathbb{R} \\ x_1, ..., x_k \in S \cup T \end{array} \right\}$$

*Note that there exists a finite collection of objects $\{y_1, ..., y_l\} \in S^l$ such that for every $x_i$ we can find a set of real numbers $\left\{\alpha_1^i, ..., \alpha_l^i\right\} \in \mathbb{R}^l$ such that*

$$x_i = \sum_{j=1}^{l} \alpha_j^i y_j$$

*This is because either $x_i \in S$, in which case we can include $x_i$ in the set $\{y_1, ..., y_l\}$, or $x_i \in T$, and so is a linear combination of elements in $S$.*

*Thus*

$$\begin{aligned} x &= \sum_{i=1}^{k} c_i \left( \sum_{j=1}^{l} \alpha_j^i y_j \right) \\ &= \sum_{j=1}^{l} \left( \sum_{i=1}^{k} \alpha_j^i c_i \right) y_j \\ &\in span(S) \end{aligned}$$

*You should check that you agree with the second line here.* ■

An obvious corollary to this is that if a set $S$ is linearly dependent, then there is a subset of $S$ which has the same span. In fact, this result is if and only if:

**Corollary 1** *Let $S \subset V$ be a subset of the linear space $V$. Then there exists a strict subset $T \subset S$ such that $span(S) = span(T)$ if and only if $S$ is linearly dependent.*

**Proof.** Left as exercise. ■

One obvious question is: what do we know about sets of objects that span the entire linear space $V$? In particular, how many objects do we need to span $V$? A minimal spanning set for a linear space $V$ is called a **basis**.

**Definition 6** *A basis for a linear space $V$ is a $\supseteq$-minimal subset of $V$ that spans $V$. In other words, $S$ is a basis of $V$ if*

1. *$span(S) = V$*

2. *If $span(T) = V$ then $T \subset S$ is false*

Thus, in some sense, a basis of $V$ is the smallest possible set that spans $V$.

This implies that any basis for $V$ must be linearly independent:

**Corollary 2** *A subset $S$ of a linear space $V$ is a basis for $V$ if and only if $S$ is linearly independent, and $span(S) = V$*

In fact, the relationship between the span and linear dependence goes deeper than that.

**Theorem 2** *If a collection of objects $S = \{x_1, x_2, .., x_n\}$ is a basis of $V$, and $\{y_1, ..., y_m\}$ are linearly independent vectors in $V$, then it must be the case that $m \leq n$*

***Proof.*** *As $S$ is a basis of $V$, we know that, for some collection of scalars $\{c_1, ..., c_n\}$*

$$y_1 = \sum_{i=1}^{n} c_i x_i$$

*WLOG assume that $c_1 \neq 0$ (why can we assume this?). Then*

$$x_1 = \frac{1}{c_1} y_1 - \sum_{i=2}^{n} \frac{c_i}{c_1} x_i$$

*Which implies that $span\{y_1, x_2, .., x_n\} = V$*

*Now, we know that $\{y_1, y_2\}$ are linearly independent. We also know that there exists a set of scalars $\{d_1, ..., d_n\}$ such that*

$$y_2 = d_1 y_1 + \sum_{i=2}^{n} d_i x_i$$

*Again, we can assume that $d_2 \neq 0$ (why), so repeating the above trick we know that $span\{y_1, y_2, .., x_n\} = V$. Iterating, we get that $span\{y_1, y_2, .., y_n\} = V$.*

*By contradiction, assume that $m > n$. Then $\{y_1, y_2, .., y_n, y_{n+1}\}$ is linearly independent, but we know that span$\{y_1, y_2, .., y_n\} = V$, so $y_{n+1}$ can be expressed as a linear combination of $\{y_1, y_2, .., y_n\}$, a contradiction.* ■

An immediate corollary to this is the following

**Corollary 3** *Any two bases of a linear space $V$ must have the same number of elements.*

Of course, a basis is not generally unique. In $\mathbb{R}^2$ any two linearly independent vectors span $\mathbb{R}^2$. By convention, we call $(1,0)$, $(0,1)$ to be the standard basis of $\mathbb{R}^2$. More generally, let the $n$ length vector $e^i$ take the value 0 in every position apart from $i$, where it takes the value 1. The collection $e^1, ..., e^n$ is the standard basis for $\mathbb{R}^n$ (you should check that this is actually basis).

The size of the bases of a linear space is one of its important properties - we call it the **dimension** of the space.

**Definition 7** *If a linear space $V$ has a finite basis $S$ then we say that $V$ is finite dimensional, and has dimension* $\dim(V) = |S|$. *If $V$ has no finite basis, then we say that it is infinitely dimensional, and* $\dim(V) = +\infty$

The idea of dimensionality is well defined by corollary 3. The dimension of a linear space tells us something about its size, in the sense that the dimension is the largest number of linearly independent objects that one can find in that linear space. In fact, any linearly independent subset of a linear space of size equal to the dimension of that space is a basis.

Thus, it is clear that $\dim(\mathbb{R}^n) = n$. There are lots of examples of infinite dimensional spaces (and in some ways this is where the mechanics that we are building here become useful). One obvious infinite dimensional space is $V = \mathbb{R}^{[0,1]}$. Another is the family of bounded functions on some space $T$ where $|T| = \infty$

Drawing together the previous results we have on spans and linear independence, we can make the following claim:

**Claim 3** *Let $V$ be a linear space, $\{x_1, ..., x_n\} \in V$ and* $\dim(V) = n$. *The following statements are equivalent:*

1. *$\{x_1, ..., x_n\}$ is linearly independent*

2. *$span(\{x_1, ..., x_n\}) = V$*

3. *$\{x_1, ..., x_n\}$ is a basis for $V$*

One final important property of a basis of a linear space is that it allows a unique *representation* of that space, as defined by the following theorem:

**Theorem 4** *Let $V$ be a linear space and $\{x_1, ..., x_n\}$ be a basis for $V$. Then every $x \in V$ has a unique representation of the form*

$$\begin{aligned} x &= c_1 x_1 + ... + c_n x_n \\ \{c_1, ..., c_n\} &\in \mathbb{R}^n \end{aligned}$$

***Proof.*** *That every $x$ can be represented in this way follows from the definition of a basis. The uniqueness part can be shown by contradiction. Say there exist two vectors*

*$\{c_1, ..., c_n\}, \{b_1, ..., b_n\} \in \mathbb{R}^n$ such that*

$$\begin{aligned} x &= c_1 x_1 +, ..., + c_n x_n \\ x &= b_1 x_1 +, ..., + b_n x_n \end{aligned}$$

*This implies that*

$$\begin{aligned} &\emptyset \\ &= x - x \\ &= \sum_{i=1}^{n} (c_i - b_i) x_i \end{aligned}$$

*But as $\{x_1, ..., x_n\}$ is a basis, it must be linearly independent, and so the only way for this to be true is for $(c_i - b_i) = 0 \ \forall \ i$, implying $c_i = b_i \ \forall \ i$* ■

## 2.2 Application: Arbitrage Pricing

Let's consider a world in which there are $l$ **commodities**. We can think of these commodities as being defined by different physical characteristics, different delivery dates, or as delivering goods in different states of nature. A **commodity bundle** is an $l$-tuple of commodities $\{x_1, ..., x_l\}$, where $x_i$ represents the amount of good $i$ in the bundle. If we allow for negative as well as positive quantities of each good, then the **commodity space** $X$ (i.e. the set of commodity bundles) is a linear space.

A **security** is a claim on a commodity bundle (equivalently, we can just think of them as commodity bundles that can be traded). Thus, we can represent a security as an $l-$length vector. Let's say that there are $n$ securities. A **portfolio** is represented by a vector $\theta \in \{\theta_1, ..., \theta_n\} \in \mathbb{R}^n$ representing the amount of each security owned (note that these can be either positive or negative). Thus, a portfolio is a claim to a commodity bundle

$$x = \sum_{j=1}^{n} \theta_j y_j$$

**Remark 2** *The set of feasible commodity bundles generated by a set of securities $Y = \{y_1, ...y_n\}$ is the span of $Y$. Thus, in order for a set of securities to span the commodity space, it must be the case that $Y$ contains a basis for $X$.This in turn implies that 'complete markets' (i.e the possibility to construct any commodity bundle) require there to be $l$ linearly independent securities.*

Say that these securities can be bought and sold on the market. The price of security $j$ is $q_j$, and let $q = \{q_1, ..., q_n\} \in \mathbb{R}^n$ be the set of security prices. Thus, the value of a portfolio is given by $\sum_{j=1}^{n} \theta_j q_j$. We say that an arbitrage exists at the price vector $q$ if there exists a portfolio $\theta$ such that

$$\begin{aligned} \sum_{j=1}^{n} \theta_j q_j &> 0 \\ \sum_{j=1}^{n} \theta_j y_j &= 0 \end{aligned}$$

In other words, an arbitrage means that there is a portfolio that gives a net payoff of 0 of each commodity, but has positive value. Obviously, if such a portfolio existed, then one could make an infinite amount of risk free profit. If no such commodity exists, then we say that the price vector $q$ satisfies the **no arbitrage condition**

**Remark 3** *An arbitrage opportunity can only exist if the set of securities is linearly dependent - in other words there exist some* ***redundant*** *securities that are linear combinations of other securities*