Mathematics For Economists

Mark Dean

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Question 1 Let $L : \mathbb{R} \to \mathbb{R}$ be a linear functional on $\mathbb{R}$. Is it necessarily the case that L maps open sets to open sets? If so, prove it. If not, provide a counterexample, and a condition on L such that the statement is true. What about if L is a linear operator mapping $\mathbb{R}^2$ to $\mathbb{R}^2$.(hint, what must $L(\mathbb{R}^2)$ look like?

Question 2 Recall that we say that a preference relation $\succeq$ (not necessarily complete) on a set X is **upper semi continuous** if the set

$$L_{\succ}(x) = \{y \in X | x \succ y\}$$

is open for every $x \in X$ (where $\succ$ is the asymmetric part of $\succeq$). It is **lower semi continuous** if the set

$$U_{\succ}(x) = \{y \in X | y \succ x\}\}$$

is open for every $x \in X$

1. Characterize the class of preference relations that are upper semi continuous in the discrete topology.
2. In $\mathbb{R}^n$ with the standard topology, give an example of a complete preference relation that is upper semi continuous but not lower semi continuous, and visa versa.
3. Is the following preference relation on $\mathbb{R}^2$ either upper or lower semi-continuous?

$$\begin{aligned} x &\succcurlyeq y \text{ iff } x_1 \geq y_1 \\ \text{and } x_2 &\geq y_2 \end{aligned}$$

what about

$$\begin{aligned} x &\succcurlyeq y \text{ iff } x_1 > y_1 \\ \text{and } x_2 &> y_2 \end{aligned}$$

?

what about

$$\begin{aligned} x &\succcurlyeq y \text{ iff } x_1 > y_1 \\ \text{or } x_1 &= y_1 \\ \text{and } x_2 &> y_2 \end{aligned}$$

4. We say a real valued function f on $\mathbb{R}$ is upper semi continuous if, for every $\alpha \in \mathbb{R}$, the set $\{x \in \mathbb{R} | f(x) < \alpha\}$ is an open set. Show that a function is upper semi continuous if and only if, for every x and every $x_n \to x$,

$$\lim \sup_{x_n \to x} f(x_n) \leq f(x)$$

Question 3 Consider the following problem:

A farmer owns a field. She is must decide what area of that field to enclose to graze sheep. The cost of 1 meter of fence is 1. For simplicty, let's assume that she can only enclose rectangular areas of land. The more land she encloses, the more sheep she can graze, and she receives income of 2 for every sheep that she grazes on her land. However, by enclosing the land, she takes away land that could otherwise be used for camping by her friends that sometimes come from out of town. These friends have a favourate camping spot, and derive utility based on the distance they are from this favourate spot. Unfortunately, the farmer doesn't know what the favourate spot is. The famer cares about money, but also the happiness of her friends, so her utility is an additive function of her income and the utility of her friends

Your job is to write down the farmer's optimization problem, and put conditions on the various elements of the problem in order to guarantee a solution. Here are some things that you might find useful. (1) How does the choice set of the friends change with the enclosure chosen by the farmer? And so how does their utility change?.[1] (2) You can use the fact that the expectation operator is continuous (3) rather than thinking of a fixed relationship between

[1] For this you may find it useful to note that, if you define a closed rectangle r in $\mathbb{R}^2$, and let O be an open set

area and number of sheep, how about a regime in which you can put as many sheep as you like in a given area, but the more sheep you put in the more likely they are to die.

Question 4 Which fixed point theorem would you use to show the following? Use the relevant theorem to prove the result

1. Let p be a vector of non-negative prices in $\mathbb{R}^n$. If firms expect price vector p^* then they will produce a vector of outputs $S(p^*) \in \mathbb{R}^n$, where S is continuous function. For a vector of outputs, then prices will be determined by $D^{-1}(q)$, where D^{-1} is the continuous inverse demand function. There is a rational expectations equilibrium of this system (assume that supply and demand are homogeneous degree 1)
2. Any game with 2 players and 2 strategies for each player has a nash equilibrium in mixed stategies .

that contains r, then any small perturbation in r will also be contained in O (if you want to make this precise, think of r as being defined by the co-ordinate of its top left and bottom right corner, then there is an epsilon ball around these points such that, any rectangle defined by points in these epsilon balls is still contained in O)