Midterm[1]

Question 1.-

Consider the set of functions $F \subset \mathcal{C}(0,1)$ defined by

$$F = \left\{ f | \mathcal{C}(0,1) | f(x) = ax^b, a \in A \subset \mathbb{R} \text{ and } b \in B \subset \mathbb{R} \right\}$$

That is, F is a subset of the set of continuous functions defined on $(0,1)$

Part 1

Let $B = \mathbb{R}$. Is every element of this set of functions continuous? Are they all uniformly continuous? Are they all Lipschitz continuous (HINT: for the last part you can use the fact that $\frac{f(x)-f(y)}{x-y}$ converges to $f'(x)$ as y converges to x)

Every element of the set of functions is continuous

Continuity of the set F follows from the following claim

Claim 1 *$f(x) = x^b$ is continuous for all $x \in (0,1)$ and $b \in \mathbb{R}$*

Proof. We are going to prove for the case in which $b \geq 0$. The case for $b < 0$ follows from the fact that $x^b = \left(\frac{1}{x}\right)^{-b}$ and from the continuity of the composition of functions and of the function $g(x) = \frac{1}{x}$ for all $x \in (0,1)$.

Let $b > 0$, the case for $b = 0$ is trivial $(f(x) = 1)$. We want to show that for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all y such that $|x - y| < \delta$ it is the case that $|x^b - y^b| < \varepsilon$. Consider $\delta \equiv \min \left\{ x - (x^b - \varepsilon)^{\frac{1}{b}}, (x^b + \varepsilon)^{\frac{1}{b}} - x \right\}$ then we have that

$$\begin{aligned}
|x - y| < \delta \quad &\Leftrightarrow \quad -\delta < y - x < \delta \\
&\Leftrightarrow \quad (x^b - \varepsilon)^{\frac{1}{b}} - x < y - x < (x^b + \varepsilon)^{\frac{1}{b}} - x \\
&\Leftrightarrow \quad (x^b - \varepsilon)^{\frac{1}{b}} < y < (x^b + \varepsilon)^{\frac{1}{b}} \\
&\Leftrightarrow \quad x^b - \varepsilon < y^b < x^b + \varepsilon \\
&\Leftrightarrow \quad -\varepsilon < y^b - x^b < \varepsilon
\end{aligned}$$

the desired result. ■ **Aside note:** The way to get to δ is to start from the bottom and work your way up.

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Not all the elements in the set are uniformly continuous

In particular consider $a = 1$ (for simplicity) and $b = -1$, the function $f(x) = \frac{1}{x}$ is not uniformly continuous. To see this take $\varepsilon = 1$ and assume by contradiction that the function is uniformly continuous, then there exist a $\delta > 0$ such that $f(B(x,\delta)) \subset B(f(x),1)$. Then it has to be the case that $\left|\frac{1}{x} - \frac{1}{y}\right| < 1$ for all x, y such that $|y - x| < \delta$.

$\left|\frac{1}{x} - \frac{1}{y}\right| < 1 \Leftrightarrow |x - y| < xy$. Set $y = x + \frac{\delta}{2}$ then $\left|\frac{\delta}{2}\right| < x^2 + x\frac{\delta}{2}$ for all $x \in (0,1)$ which is clearly not true.

Not all the elements in the set are Lipstchiz continuous

Consider $a = 1$ and $b = \frac{1}{2}$, and assume by the way of contradiction that the function is Lipschitz continuous. Then it is the case that there exists a $k \in \mathbb{R}_+$ such that $|f(x) - f(y)| \leq K|x - y|$ or equivalently (assuming $x \neq y$) $\frac{|f(x)-f(y)|}{|x-y|} \leq K$. Consider the case where $y = 2x$ then it must be the case that

$$\begin{aligned}
\frac{|\sqrt{x} - \sqrt{2x}|}{|x - 2x|} \leq K \quad &\Leftrightarrow \quad \frac{|\sqrt{x}(1 - \sqrt{2})|}{|-x|} \leq K \\
&\Leftrightarrow \quad \frac{|(1 - \sqrt{2})}{\sqrt{x}} \leq K
\end{aligned}$$

Consider now $x \to 0$ then the last inequality clearly is not true.

Part 2.-

Does your answer change if we instead have $B = \mathbb{R}_{++}$

The function is clearly still continuous from the previous part and it is not Lipschitz continuous since the counterexample of part 1 still applies.

In this case the of uniformly continuity now it is. I think the easiest way to prove it is as follows. Consider first the family of function of the form $f(x) = ax^b$ but when $[0,1]$ and then conclude that is uniformly continuous in $(0,1)$. In particular we just need to prove that is continuous in $[0,1]$ since as it was proving in the homework continuity of a function on $[a,b]$ implies uniformly continuity on $[a,b]$. In particular we just need to prove continuity at $x = 0$ and $x = 1$, the rest follows from the previous part.

Continuity of $x = 0$

We want to show that for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $y \in B(0,\delta)$ it is the case that $f(y) \in B(f(0),\varepsilon)$. Again for simplicity consider the case where $a = 1$ and consider

$\delta = (\varepsilon)^{\frac{1}{b}}$

$$\begin{aligned}
&\Leftrightarrow \quad -(\varepsilon)^{\frac{1}{b}} < y - 0 < (\varepsilon)^{\frac{1}{b}} \\
&\Leftrightarrow \quad -(\varepsilon)^{\frac{1}{b}} < y < (\varepsilon)^{\frac{1}{b}} \\
&\Leftrightarrow \quad -\varepsilon < y^b < \varepsilon \\
&\Leftrightarrow \quad -\varepsilon < y^b - 0 < \varepsilon \\
&\Leftrightarrow \quad -\varepsilon < f(y) - f(0) < \varepsilon
\end{aligned}$$

Continuity of $x = 1$

We want to show that for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $y \in B(1, \delta)$ it is the case that $f(y) \in B(f(1), \varepsilon)$. Again for simplicity consider the case where $a = 1$. Consider $\delta = \min\left\{1 - (1-\varepsilon)^{\frac{1}{b}}, (1+\varepsilon)^{\frac{1}{b}} - 1\right\}$

$$\begin{aligned}
&\Leftrightarrow \quad -(\varepsilon - 1)^{\frac{1}{b}} - 1 < y - 1 < (\varepsilon + 1)^{\frac{1}{b}} - 1 \\
&\Leftrightarrow \quad -(\varepsilon - 1)^{\frac{1}{b}} < y < (\varepsilon + 1)^{\frac{1}{b}} \\
&\Leftrightarrow \quad -\varepsilon + 1 < y^b < \varepsilon + 1 \\
&\Leftrightarrow \quad -\varepsilon < y^b - 1 < \varepsilon \\
&\Leftrightarrow \quad -\varepsilon < f(y) - f(1) < \varepsilon
\end{aligned}$$

Now assume that $B \subset \mathbb{R}_{++}$ and that $F \subset \mathcal{C}[0,1]$ which we endow with the sup metric (i.e. $d(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|$)

Part 3.-

Can we ensure that the distance between any two functions in F is finite?

Yes, because if $b \in \mathbb{R}_{++}$ and $x \in [0,1]$ then $f(x) \in [0, a]$ if $a \geq 0$ and $f(x) \in [a, 0]$ if $a < 0$ for all $x \in [0,1]$ and for all $a \in \mathbb{R}$ and $b \in \mathbb{R}_{++}$ and if the functions are bounded then the distance is finite, see homework.

Part 4.-

Is F bounded? If not, provide necessary and sufficient conditions on A and B for it to be bounded?

F is bounded if there exists a r and a function f such that $F \subset B(f, r)$. In particular we need A to be bounded, if A is bounded then $d(f_1, f_2) = \sup_{x\in[0,1]} |f_1(x) - f_2(x)| = \sup_{x\in[0,1]} |a_1 x^{b_1} - a_2 x^{b_2}| \leq |\sup A - \inf A| < \infty$. Consider the function $f(x) = ax^b$ when $a = 0$ then $F \subset B(0, R)$ where $R \equiv |\sup A - \inf A| + 1$.

Part 5.-

Is F complete? If not, provide sufficient conditions on A and B for it to be complete (you will get more points if you can name looser conditions!)

The set is not complete. Consider for example the following Cauchy sequence $x^{\frac{1}{m}} \in F$ for all m and $x \in [0,1]$. Then $f_m \to f$ where $f(0) = 0$ and $f(x) = 1$ for all $x \in (0,1]$ which is clearly not continuous and $f \notin F$ ($b \in \mathbb{R}_{+}+$).

For F to be complete we need that any cauchy sequence in F is convergent in F. Take any cauchy sequence $f_n \in F$ we want to show (or find the conditions under which) $f_n \to f \in F$. If f_n is cauchy then it is the case that for all $\varepsilon > 0$ there exists a N such that if $n, m > N$ then $d(f_n, f_m) < \varepsilon$ that is $\sup_{x\in[0,1]} |a_n x^{b_n} - a_m x^{b_m}|$. In case that A is a singleton then we need completeness of B and the other way around, follows from continuity.

More generally consider f_n Cauchy, that means that for every $\varepsilon > 0$ there exists some $M \in \mathbb{N}$ such that $d(f_n, f_m) < \varepsilon$ for all $m, n > M$; that is $\sup_{x\in[0,1]} |f_n(x) - f_m(x)| < \varepsilon$ or $\sup_{x\in[0,1]} |a_n(x)^{b_n} - a_m(x)^{b_m}| < \varepsilon$. In particular when $x = 1$ f_n being Cauchy implies a_n being cauchy, and therefore one of the conditions that we must impose is the completeness of A.

Furthermore if a_n is Cauchy then for f_n to be Cauchy b_n must be Cauchy (following a continuity argument) and therefore in order to get completeness of F we need completeness of B.

Part 6.-

Define $T : F \to \mathbb{R}$ as $T(f) = \max_{x\in[0,1]} f(x)$. Is T well defined (i.e. does $T(f)$ take a value in $\mathbb{R}$ for any $f \in F$?)

Yes, by Weierstrass since $[0,1]$ is a compact space and f is a continuous function then there exists a maximum.

Part 7.-

Provide conditions under which the problem $\max_{f\in F} T(f)$ has a solution (again, you will get more points for looser conditions!)

In order to be sure that the problem has a solution we have to see the conditions under which we can ensure $T(f)$ to be continuous and F to be compact.

We know that the set F is compact if it is closed, bounded and equicontinuous. For boundedness we need boundedness of A. For closeness take any sequence of $f_m \in F$ such that $f_m \to f$ we need to show (or find the conditions under which) $f \in F$. By imposing A and B closed we get the desired result

F is equicontinuous at $x \in [0,1]$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all y such that $|x-y| < \delta$ implies $|ax^b - ay^b| < \varepsilon$ for all $a \in A$ and $b \in B$. From previous parts we have that A is closed and bounded and B is closed. Considering $a = 1$ and $b = m$ we can show that F is not equicontinuous when $m \to \infty$, see homework 3. Then we need to imposed boundedness on B.

Putting all these together, $A \subset \mathbb{R}$ and $B \subset \mathbb{R}_{++}$ must be compact.

In order to prove that the function $T(f)$ is continuous we can invoke the theorem of the maximum. In particular prove that the function f is continuous and we know that $[0,1]$ is a compact set, then we know that the set of maximizers is an upper hemicontinuous correspondence. Also given that $b \in \mathbb{R}_{++}$ then we have that the function $f(x) = ax^b$ is strictly monotonic if $a \neq 0$. Then the maximizer is unique

Question 2 .-

Let X be a metric space such that there exists an $\varepsilon > 0$ and an uncountable set $S \subseteq X$ such that $d(x,y) > \varepsilon$ for any distinct $x, y \in S$. Show that X cannot be separable. Give an example of three spaces that satisfies this property.

A metric space X is separable if it has a countable dense subset. Let assume by contradiction that X is separable and let Y be the countable dense subset, that is $X = cl(Y)$ and Y is countable. For any $x \in S$, then it has to be the case that $x \in Y$ or $x \in cl(Y) \setminus Y = X$. We know it has to be the latter because by assumption Y is countable. Consider $\varepsilon' = \frac{1}{2}\varepsilon$ then $B(x, \varepsilon') = x \cup Y = \emptyset$ which contradicts that $x \in cl(Y)$.

Examples:

- $\mathbb{R}$ with the discrete metric.
- l^∞
- Hedgehog space with uncountable spinyness

Question 3

Consider the following subsets of the set of all infinite sequences

$$l^p = \left\{ \{x\}_m \in \mathbb{R}^\infty | \sum_{i=1}^{\infty} |x_i|^p < \infty \right\}$$

$$l^\infty = \left\{ \{x\}_m \in \mathbb{R}^\infty | \sup_{m \in \mathbb{N}} |x_m| < \infty \right\}$$

Part 1.-

Show that l^1 is a strict subset of l^∞

We want to show that (1) $\{x\}_m \in l^1 \Rightarrow \{x\}_m \in l^\infty$ and that (2) $l^\infty \setminus l^1 \neq \emptyset$

(1) $l^1 = \{\{x\}_m \in \mathbb{R}^\infty | \sum_{i=1}^\infty |x_i| < \infty\}$. In particular for $\sum_{i=1}^\infty |x_i| = M < \infty$ it must be the case (as it was shown in class) that $\lim_{i\to\infty} |x_i| = 0$. As it was also shown in class any convergent subsequence is bounded, therefore $\sup_{m\in\mathbb{N}} |x_m| < \infty$

(2) For example $\{x\}_m = \frac{1}{m} \in l^\infty$ ($\sup_{m\in\mathbb{N}} |x_m| = 1$) but $\{x\}_m = \frac{1}{m} \notin l^1$ ($\sum_{i=1}^\infty |x_i| = \infty$)

Part 2.-

Is l^1 a linear subspace of l^∞?

l^1 is a linear subspace of l^∞ (from previous part we know that is a subset) if it is closed under addition and scalar multiplication. In particular consider the sequences $\{x\}, \{y\} \in l^1$ we need to show that $\{z\} = \{x + y\} \in l^1$ and that $\{w\} = \{\lambda x\} \in l^1$ for $\lambda \in \mathbb{R}$.

$$\{z\} \in l^1 \Leftrightarrow \sum_{i=1}^{\infty} |z_i| < \infty \Leftrightarrow \sum_{i=1}^{\infty} |x_i + y_i| < \infty$$

since $|x_i + y_i| \leq |x_i| + |y_i|$ for all i then it is the case that $\sum_{i=1}^\infty |x_i + y_i| \leq \sum_{i=1}^\infty |x_i| + |y_i| = \sum_{i=1}^\infty |x_i| + \sum_{i=1}^\infty |y_i| < \infty$ where the last inequality follows from the fact that $\{x\}, \{y\} \in l^1$.

$$\begin{aligned} \{w\} \in l^1 &\Leftrightarrow \sum_{i=1}^{\infty} |w_i| < \infty && (1)\\ &\Leftrightarrow \sum_{i=1}^{\infty} |\lambda x_i| < \infty && (2)\\ &\Leftrightarrow |\lambda| \sum_{i=1}^{\infty} |x_i| < \infty && (3) \end{aligned}$$

where the last inequality follows from the fact that $\{x\} \in l^1$

Part 3.-

Is the set of all convergent real sequences a linear subspace of l^∞? or of l^1?

As it was shown in class the set of all convergent real sequences is a subset of l^∞, but it is not a subset of l^1, consider for example the sequence $x_m = \frac{1}{m}$ which is convergent but the infinite sum is infinite. Now we need to show that the set of convergent subsequence is a linear subspace of l^{∞}. As in the previous part, we need to prove that the set of convergent subsequences is closed under addition and scalar multiplication. Which follows directly from the fact that if $x_n \to x$ and $y_n \to y$ then $x_n + y_n \to x + y$ and $\lambda x_n \to \lambda x$.