2 Lecture 2

2.1 Closed Sets

Along with the notion of openness, we get the notion of closedness.

Definition 6 *Let M, d be a metric space, then a set $S \subset M$ is closed if M/S is open*

In $\mathbb{R}$, closed intervals are closed (as we might hope). To see this, note that

$$\mathbb{R}/[\alpha, \beta]$$

$$(-\infty, \alpha) \cup (\beta, \infty)$$

which is the union of two open sets (and therefore open).

It is also true that, in any metric space M, d, a singleton $\{x\} \subset M$ is closed. to see this, we need to show that $M/\{x\}$ is open. But to show this, take any $y \in M/\{x\}$, define $r = d(x, y)$, which is greater than zero by the definition of a metric and note that $x \notin B(y, r)$, showing that $M/\{x\}$ is open.

A direct corollary of the definition of a topology and the definition of closed sets (which you should check) is the following properties of closed sets

Corollary 1 *Let X be an arbitrary set with some topology T*

1. *The empty set and X are closed*
2. *If $\{F_\alpha\}$ is a collection of closed sets, then $\cap_\alpha F_\alpha$ is closed*
3. *If $\{F_\alpha\}$ is a finite collection of closed sets, then $\cup_\alpha F_\alpha$ is closed*

Note that the empty set and X are both closed and open, a property we call **clopen.**

A **closure point** of a set S is a point x that is 'close' to the set, in the sense that for every open ball $B(x, r)$, there is some element of S in that ball.

Definition 7 *For any set S, a closure point of S is a point x such that, for any $r > 0$*

$$B(x, r) \cap S \neq \varnothing$$

This leads to another characterization of closed sets:

Lemma 2 *A set F is closed in a space M if and only if it contains all its closure points*

Proof. *Assume F is closed. Let x be a closure point. Then there is no r such that $B(x, r) \in M/F$. Thus, if $x \in M/F$, then M/F is not open, so F would not be closed.*

Now say that F contains all its closure points Let $x \in M/F$. Then there must exist some r such that $B(x, r) \cap F = \varnothing$ (otherwise x is a closure point). Thus, M/F is open, and so F is closed.

■

Another characterization of closed sets that is going to come in very handy involves the concept of convergent sequences. In order to explain this, we need to explain what we mean by a convergent sequence. First of all, we should remember that a sequence in a particular set is just a mapping from the natural numbers to elements of that set, which we write $\{x_1, x_2, ..., \}$ or $\{x_i\}_{i=1}^{\infty}$. A **convergent** sequence is a sequence that gets closer and closer to a particular point in a metric space

Definition 8 *A sequence $\{x_i\}_{i=1}^{\infty}$ in a metric space M converges to a point $x \in M$ if, $\forall\ r > 0$, $\exists$ N such that $x_n \in B(x, r)\ \forall\ n \geq N$. In this case, we write $\{x_i\}_{i=1}^{\infty} \to x$, and call x the limit of $\{x_i\}_{i=1}^{\infty}$, which we also write as $\lim_{i \to \infty} x_i = x$.*

Thus, for x to be the limit of $\{x_i\}_{i=1}^{\infty}$ it has to be the case that for any ball around x, we can find a point such that, after that point, the sequence never leaves that ball.

Just to get a handle on the concept of convergence, let's consider a couple of examples

Example 3 *Any constant sequence converges. Moreover, any sequence that finally becomes constant converges*

Example 4 *Under the Euclidian metric, a sequence $\{x_1, ..., \} \in \mathbb{R}^n$ converges if and only if $\{(x_i), ...\} \in \mathbb{R}$ converges*

We will come back to some of the properties of sequences and limits later. For now, what we need to know is that, a set is closed if and only if any convergent sequence in that set converges to a limit in that set. This is going to come in very handy

Lemma 3 *A set F is closed in some metric space M if and only if $\{x_i\}_{i=1}^{\infty} \subset F$ and $\{x_i\}_{i=1}^{\infty} \to x$ implies that $x \in F$*

Proof. *Assume F is closed, $\{x_i\}_{i=1}^{\infty} \subset F$ and $\{x_i\}_{i=1}^{\infty} \to x$. This implies that, for any $r > 0$, $\exists\ n$ such that $x_n \in B(x, r)$. As $x_n \in F$, x is a closure point F, and by lemma 2 $x \in F$. Next, assume that every convergent sequence converges to a point in F. Let x be a closure point of F, then, for every $r > 0$, $B(x, r) \cap F \neq \varnothing$. But this means that we can construct a sequence $\{x_i\}_{i=1}^{\infty}$ and a sequence of balls $B(x, \frac{1}{i})$ such that $x_i \in B(x, \frac{1}{i})$.[1] Clearly, $\lim_{i \to \infty} x_i = x$, thus $x \in F$. This tells us that F contains all its closure points, so again by lemma 2 is closed.* ■

Related to the idea of closed and open sets are the concepts of the **closure** and **interior points** of a set

Definition 9 *The closure of a set F (denoted $cl(F) = \bar{F}$) is the smallest closed set that contains F*

$$\bar{F} = \cap \{S | F \subset S \text{ and } S \text{ is closed}\}$$

The interior of a set F (denoted $int(F) = \mathring{F}$) is the union of all open sets that are contained by F

$$\mathring{F} = \cup \{S | S \subset F \text{ and } S \text{ is open}\}$$

The boundary of a set F (denoted $bdry(F)$) is the intersection of the closure of F and its compliment

$$bdry(F) = cl(F) \cap cl(M/F)$$

It should be immediately obvious that a set F is closed if and only if $F = cl(F)$ and open if and only if $int(F) = F$. Less obvious properties that you will prove for homework are given as follows

Proposition 1 *Properties of closures and interior points*

[1] For those of you that are worried about such things, I am using the axiom of choice here

1. $cl(A) = \{x \in M | x \text{ is a closure point of } A\}$

2. $int(A) = \{x \in M | \exists\ r > 0 \text{ such that } B(x, r) \subset A\}$

3. *If* $A \subset B$ *then* $cl(A) \subseteq cl(B)$*. However, if* $cl(A) \subseteq cl(B)$ *then it is not necessarily the case that* $A \subseteq B$

4. *For* A,$B \subset M$*, then* $cl(A \cup B) = cl(A) \cup cl(B)$ *and* $cl(A \cap B) \subseteq cl(A) \cap cl(B)$ *but it is not necessarily the case that* $cl(A \cap B) = cl(A) \cap cl(B)$

To fix ideas, consider the following example

Example 5 *Consider the set* $A = [0, 1] \cap \mathbb{Q}$ *then*

1. $cl(A) = [0, 1]$

2. $int(A) = \{\}$

3. $bdry(A) = [0, 1]$

Proof. *All of this follows from the fact that, for any real number* x *and* $r > 0$*,* $B(x, r) \cap \mathbb{Q} \neq \{\}$ *and* $B(x, r) \cap \mathbb{R}/\mathbb{Q} \neq \{\}$*. (If you don't understand why this is true, we should go through it). This implies that, for every* $x \in [0, 1]$ *and* $r > 0$*,* $B(x, r) \cap A \neq \{\}$*, and so* $x \in cl(A)$*. Similarly, for every* $x \in A$*,* $r > 0$*,* $B(x, r) \cap \mathbb{R}/\mathbb{Q} \neq \{\}$*, and so* $x \neq int(A)$*. Finally, this also implies that* $B(x, r) \cap A/\mathbb{R} \neq \{\}$*, and so* $x \in cl(A/\mathbb{R})$*, implying* $x \in bdry(A)$*.* ■