4 Lecture 4

4.1 Real Sequences

We are now going to go back to the concept of sequences, and look at some properties of sequences in $\mathbb{R}$

Definition 13 *A real sequence is increasing if $x_{n+1} \geq x_n$ for all n, and strictly increasing if $x_{n+1} > x_n$ for all n. The concepts of decreasing and strictly decreasing sequences are defined analogously. A sequence is monotone if it is either increasing or decreasing. A real sequence is bounded if there exists $L, U \in \mathbb{R}$ such that $L < x_n < U \ \forall \ n$*

The first property of real sequences is that, a sequence that is monotone and bounded must eventually converge

Lemma 5 *A monotone bounded sequence of real numbers converges*

Proof. *WLOG, assume that x_n is increasing, and let $x = \sup x_n$. Our claim is that $x_n \to x$.*

For any $\varepsilon > 0$, by the definition of $\sup x_n$ there exists x_n such that $|x_n - x| < \varepsilon$. Set $N = n$. Then, by the monotonicity $x_m > x_N \ \forall \ m > n$. But as $x_m < x$, this implies that $|x_m - x| < \varepsilon \ \forall \ m > N$, so $x_n \to x$ ■

In order to understand the next property, we need to define the concept of a subsequence

Definition 14 *A subsequence of a sequence x_n is a sequence y_n such that there exists a function $f : \mathbb{N} \to \mathbb{N}$ strictly increasing such that $y_i = x_{f(i)} \ \forall \ i \in \mathbb{N}$*

It turns out that every sequence of real numbers has subsequence that is monotone.

Lemma 6 *Every sequence of real numbers has a monotone subsequence.*

Proof. *Let $S = \{n | x_m > x_n \ \forall \ m > n\}$. If S is infinite $\{n_1, n_2, ..\}$, then the sequence $x_{n_1} < x_{n_2} < x_{n_3} < ...$ is a monotone subsequence.*

If S is finite, then $\exists\ n_1$ such that $n < n_1\ \forall\ n \in S$. Since $n_1 \notin S$, then $\exists$ a $n_2 > n_1$ such that $x_{n_2} \leq x_{n_1}$. As $n_2 \notin S$, $\exists\ n_3$ such that $x_{n_3} \leq x_{n_2} \leq x_{n_1}$ and so on. ■

An immediate corollary of these two lemmas is the Bolzano - Weierstrass theorem

Theorem 4 *(Bolzano-Weierstrass) Any bounded sequence of a real numbers has a convergent subsequence*

Any subsequence of a convergent real sequence converges to the limit of the mother sequence. (yes?) What is more, even if the mother sequence is divergent, it may still possess a convergent subsequence (as in the Bolzano-Weierstrass Theorem). This suggests that we can get at least some information about the long run behavior of a sequence by studying those points to which at least one subsequence of the sequence converges.

Definition 15 *For any real sequence x_m, we say that $x \in \bar{\mathbb{R}}$ is a subsequential limit of x_m if there exists a subsequence $x_{m_k} \to x$.*

For example, the sequence $x_m = (-1)^m$ has two subsequential limits, -1 and 1.

If x is a subsequential limit of x_m, it means that x_m visits $B(x, \varepsilon)$ infinitely often, for any $\varepsilon > 0$. This is the sense in which subsequential limits tell us something about the limiting behavior of x_m

4.2 Lim-Sup and Lim-Inf

Two subsequential limits that are of particular interest are the greatest and least subsequential limits of a sequence

Definition 16 *For any real sequence x_m we write $x = \limsup x_m$ if*

1. *For any $\varepsilon > 0$, there exists an M such that $x_m < x + \varepsilon$ for every $m \geq M$*
2. *For every $\varepsilon > 0$ and $M \in \mathbb{N}$, there exists a $k > M$ such that $x_k > x - \varepsilon$*

   We write $\limsup x_m = +\infty$ if $+\infty$ is a subsequential limit of x_m. The concept of $\liminf$ is defined analogously.

In other words, $x = \limsup x_m$ if all but finitely many terms are below $x + \varepsilon$ for any $\varepsilon > 0$, and infinitely many terms are above $x - \varepsilon$ for any $\varepsilon > 0$. Thus, the concept of the $\limsup$ is weaker than the concept of a limit, were we could also say that there were only finitely many terms below $x - \varepsilon$. Here, there could be infinitely many such terms, it is just that there also has to be infinitely many terms above $x - \varepsilon$.

In order to clarify the role of $\liminf$ and $\limsup$, it is worth going through the following properties

Remark 2 *The following are properties of* $\liminf$ *and* $\limsup$

1. $\limsup x_m = \inf(\sup\{x_n, x_{n+1}, ...,\} | n = 1, 2, ...)$
2. $\limsup x_m = \sup\{x \in \bar{\mathbb{R}} | x \textit{ is a subsequential limit of } x_m\}$
3. *Every real sequence has a* $\liminf$ *and a* $\limsup$ *in* $\bar{\mathbb{R}}$
4. $\liminf x_m \leq \limsup x_m$
5. *Any real sequence* x_m *has a monotone real subsequence that converges to* $\limsup x_m$
6. *A sequence converges if and only if* $\liminf x_m = \limsup x_m$

Proof. *We do each claim in turn*

1. *Let* $x = \inf(\sup\{x_n, x_{n+1}, ...,\} | n = 1, 2, ...)$*.If* $x = \infty$*, then we can clearly construct a subsequence that converges to* $+\infty$*. If not, then pick some* ε*. There exists some* N *such that* $\sup\{x_N, x_{N+1}, ...,\} < x + \varepsilon$*, otherwise* x *is not the largest lower bound of that set. Thus,* $x_n < x + \varepsilon$ *for* $\forall\ n \geq N$*. Moreover, for any* M*,* $\sup\{x_M, x_{M+1}, ...,\} \geq x$*. Thus, for any* $\varepsilon > 0$*, there exists a* $k > M$ *such that* $xk < x - \varepsilon$
2. *If* x_m *is unbounded above then we can construct a subsequence going to* ∞*, so clearly* ∞ *is both the sup of the set of subsequential limit and so by definition* $\limsup x_m$*. If* x_m *is not bounded below, then we can construct a subsequence going to* $-\infty$*, so* $-\infty$ *is a subsequential limit of* x_m*. Thus, by the Bolzano Weierstass theorem, the set of subsequential limits is non-empty and bounded above so it has a sup, which we will define as* x*. Now assume that for some* ε*,*

there is no M such that $x_m < x+\varepsilon$ for every $m \geq M$. Then we can construct a subsequence x_{m_k} such that $x_{m_k} \geq x+\varepsilon$. As this sequence is bounded above by assumption and below, it has a convergent subsequence. But this must converge to a subsequential limit $\bar{x} \geq x+\varepsilon$, a contradiction. Now say that for some ε and $M \in \mathbb{N}$, there exists no a $k > M$ such that $x_k > x-\varepsilon$. Then any subsequential limit $\bar{x}$ of x_m such that $\bar{x} \geq x-\varepsilon$, again a contradiction

3. *As we have shown above, every real sequence has to have subsequential limit in $\bar{\mathbb{R}}$.Then either this set is unbounded, in which case the lim sup is ∞, or it is bounded above, in which case the sup of the set of subsequential limits is well defined, and by the above proof, the lim sup.*

4. *Say $x = \liminf x_m > \limsup x_m = y$.Let $\varepsilon = \frac{\liminf x_m - \limsup x_m}{2} > 0$, then, there exists an M such that $x_m > x-\varepsilon$ for all $m > M$. But there also for any N there has $x_m < y+\varepsilon$. WLOG say $M > N$ then*

$$\begin{aligned} x_m &< y+\varepsilon \\ &= x-\varepsilon \\ &< x_m \end{aligned}$$

5. *This is trivial if $\limsup x_m = +\infty$, so assume not and that $x = \limsup x_m$ We can show that this is a subsequential limit of x_m, as we can define a sequence of balls $B(x, \frac{1}{k})$ and a subsequence x_{m_k} such that $x_{m_k} \in B(x, \frac{1}{k})$ for all k. monotonicity follows from the fact that every sequence of real numbers has a monotone subsequence*

6. *Exercise*

■

4.3 Summing Real Sequences

One final thing that we might want to do with real sequences is sum them. For example, we generally define the utility of an infinite consumption sequence in that way. Formally, we define the summation of an infinite sequence in the following way:

Definition 17 *Let $\{x_i\}_{i=1}^{\infty}$ be a real sequence. Define the sequences $\{\sum_{i=1}^{m} x_i\}_{m=1}^{\infty}$ as the sequence of finite sums up to element m. We define $\sum_{i=1}^{\infty} x_i$ as the limit of this sequence, if such a limit exists.*

Obviously, $\sum_{i=1}^{\infty} x_i$ is not defined in $\mathbb{R}$ for every sequence. For example, any constant sequence that is not equal to zero will not have $\sum_{i=1}^{\infty} x_i$ defined. In this case, the problem is that the sequence $\sum_{i=1}^{m} x_i$ either goes to $+\infty$ or $-\infty$. However, we cannot solve this problem by asking $\sum_{i=1}^{\infty} x_i$ to exist in $\bar{\mathbb{R}}$. For example, consider the sequence

$$x_i = (-1)^i$$

The sequence

$$\sum_{i=1}^{m} x_i$$

has no limit.

So what sequences have infinite sums? Well, one necessary, but not sufficient condition is that the limit of the sequence is equal to zero. To see this, note that

$$\begin{aligned}
\lim_{m\to\infty} x_m &= \lim_{m\to\infty}\left(\sum_{i=1}^{m} x_i - \sum_{i=1}^{m-1} x_i\right) \\
&= \lim_{m\to\infty}\sum_{i=1}^{m} x_i - \lim_{m\to\infty}\sum_{i=1}^{m-1} x_i = 0
\end{aligned}$$

So it is clearly necessary. To see that it is not sufficient, note that the sequence $\left\{\frac{1}{i}\right\}_{i=1}^{\infty}$ converges to zero, but $\sum_{i=1}^{m}\frac{1}{i}$ goes to infinity[3]

Here are some sequences that do have infinite sums

Example 7 $\sum_{i=1}^{\infty}\frac{1}{i^\alpha}$ *exists if* $\alpha > 1$*. To see this, note that*

$$\begin{aligned}
\sum_{i=1}^{m}\frac{1}{i^\alpha} &\leq 1 + \int_1^m \frac{1}{t^\alpha}dt = 1 + \frac{1}{1-\alpha}\left(1 - \frac{1}{m^{\alpha-1}}\right) \\
\textit{so } \lim_{m\to\infty}\sum_{i=1}^{m}\frac{1}{i^\alpha} &= 1 + \frac{1}{1+\alpha} - \frac{1}{1-\alpha}\lim_{m\to\infty}\left(\frac{1}{m}\right)^{\alpha-1}
\end{aligned}$$

and $\lim_{m\to\infty}\left(\frac{1}{m}\right)^{\alpha-1} = 0$ *if* $\alpha > 1$

[3] Consider $\{\chi_m\}$ defined as $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, ...\right)$. Note that, for every i

$$\sum_{m=1}^{2^i-1}\chi_m = \sum_{m=1}^{2^i-1}\frac{1}{2}^i$$

So this sequence does not have an infinite sum, and $\sum_{i=1}^{m}\frac{1}{i} \geq \sum_{i=1}^{m}\chi_m$

Example 8 *One result that will be useful for you is the following*

$$\sum_{i=1}^{\infty} \delta^i = \frac{\delta}{1-\delta}$$

for any $-1 < \delta < 1$

To see this, note that

$$\left(1 + \delta^1 + \delta^2 + ... + \delta^m\right)(1-\delta) = 1 - \delta^{m+1}$$

so

$$\begin{aligned} & \lim_{m\to\infty} \sum_{i=1}^{m} \delta^i \\ = & \lim_{m\to\infty} \frac{1-\delta^{m+1}}{1-\delta} - 1 \\ = & \frac{\delta}{1-\delta} \end{aligned}$$

4.4 Complete Metric Spaces

We now move on to another type of sequence

Definition 18 *Let* (M,d) *be a metric space. A sequence* $\{x_n\}$ *is Cauchy if, for every* ε*, there exists an* N *such that* $d(x_n, x_m) < \varepsilon$ *for every* $n, m > N$

Thus, a Cauchy sequence is one such that its elements become arbitrarily 'close together' as we move down the sequence. It should be fairly clear (though we will now quickly prove) that convergent sequences are Cauchy

Lemma 7 *A convergent sequence is Cauchy*

Proof. *Let* $x_n \to x$ *and pick an arbitrary* ε*. For* $\frac{\varepsilon}{2}$ *there exists an* N *such that, for any* $n, m > N$*,* $d(x_n, x) < \frac{\varepsilon}{2}$ *and* $d(x_m, x) < \frac{\varepsilon}{2}$*. Thus, by the triangle inequality,* $d(x_n, x_m) < \varepsilon$ ■

However, is it always the case that every Cauchy sequence converges? In general, no. Consider the following two example:

Example 9 *Let* $M = (0, 1]$*, and consider the sequence* $x_n = \frac{1}{n}$*. The sequence* x_n *is Cauchy (check), but does not converge to any* $x \in M$*.*

Example 10 *Let* $M = \mathbb{Q}$ *with the standard metric and consider* $x_n = 1 + \sum_{m=1}^{n} \frac{1}{m!} \to e$*. Again, this sequence is Cauchy, but does not converge in* $\mathbb{Q}$

A very important class of metric spaces are those in which Cauchy sequences are guaranteed to converge. Such spaces are called **complete**.

Definition 19 *A metric space* (M, d) *is complete if any Cauchy sequence converges to some point in* M*.*

One reason that this is a nice property is that it is often easier to check whether a sequence is Cauchy than whether it converges: in complete metric spaces we know that one implies the other. You will be genuinely shocked to find that $\mathbb{R}$ is complete with the standard metric

Theorem 5 $\mathbb{R}$ *is complete with the standard metric*

Proof. *Let* $\{x_n\}$ *be a Cauchy sequence. First, we show that it is bounded. To see this, pick* $\varepsilon = 1$*. There must be some* N *such that* $|x_m - x_n| < 1 \ \forall \ m, n \geq N$*. Let* $B = \max\{x_n | n < N\}$*, then* $x_n < B + 1 \ \forall \ n$*. Similarly, we can find a lower bound.*

As $\{x_n\}$ *is bounded, it must have a convergent subsequence* $x_{n_k} \to x$*. Our claim is that* $x_n \to x$*. To see this, note that, for any* ε*, we can find an* M *such that* $x_{n_k} \in B(x, \frac{\varepsilon}{2})$ *and* $|x_n - x_{n_k}| < \frac{\varepsilon}{2}$ *for every* $n_k, n > M$*. Thus, by the triangle inequality,* $|x_n - x| < \varepsilon \ \forall \ m \geq n$*.* ■

It follows relatively immediately that $\mathbb{R}^n$ is complete.

Those of you who are still awake will probably have spotted some apparent similarities between completeness and closedness. In fact, completeness implies closedness

Theorem 6 *Let* X *be a metric space and* Y *be a metric subspace of* X*. If* Y *is complete, then it is closed.*

Proof. *Let* Y *be complete. Take any convergent sequence* $y_m \to y$*. Then* y_m *is also Cauchy. By completeness* $\{y_m\}$ *converges in* Y*, so* Y *is closed.* ■

Closedness does not imply completeness in general, but it is true that any closed subset of a complete metric space is complete (why?). An example of a closed set that is not complete is $X = \mathbb{R}/\mathbb{Q}$, which is closed IN ITSELF (though not in $\mathbb{R}$), but is not complete. Again, the problem is that there are things 'missing' from the set, so a sequence can 'converge', but has nothing to converge to.