5 Lecture 5

5.1 Connectedness and Separability

The general class of metric spaces is large, and contains many ill behaved examples (one of which is any set endowed with the discrete metric - good for gaining intuition, a nightmare to work with). We are going to therefore introduce two regularity conditions that give us 'nice' metric spaces.

The first of these conditions is connectedness. A connected metric space is one that cannot be chopped into two open sets.

Definition 20 *A metric space is connected if there do not exist two non-empty and disjoint open sets O and U such that $M = O \cup U$. A subset S of M is connected in M if S is a connected metric subspace of M (i.e. it cannot be written as the union of two subspaces that are open in S)*

One interesting characterization of a connected metric space is that it is one in which the only clopen sets are the empty set and the whole space.

Proposition 2 *A metric space M is connected if and only if the only clopen subsets of M are the empty set*

Proof. *Let S be a clopen strict subset of M. Then $S \cup M/S = M$ and S and M/S are disjoint open subsets, meaning that M is not connected. Conversely, assume that M is not connected, then there exists two subsets O and U that are open, nonempty and $O \cup U = M$. But this implies that both O and U are clopen, as O is open, but so is $U = M/O$.* ■

This tells us directly that any set endowed with more than one object that is endowed with the discrete metric is not connected.

It is relatively easy to show that any interval in $\mathbb{R}$ is connected, and by the same argument that $\mathbb{R}$ itself is connected. In fact, this result is if and only if. Any subset of $\mathbb{R}$ that is not an interval is not connected.

For homework you will show some a nice property of continuous functions on connected metric spaces: The intermediate value theorem holds.

Next we move on to separable metric spaces. As we shall see, these are spaces that are not 'too big' in that they contain only a countable number sets.

Definition 21 *Let M be a metric space. A set Y is dense in M if $M = cl(Y)$. We say a metric space is separable if it has a countable dense subset.*

Using the fact that any point in the closure of a set is the limit of a sequence in that set (yes?) it is easy to show that $\mathbb{Q}$ is dense in $\mathbb{R}$, and so $\mathbb{R}$ is separable. A discrete metric space is separable if and only if it is countable. My favourite example of a non-separable space is a hedgehog space of uncountable spinyness.

One handy result is that the set of continuous functions defined on a closed interval is separable (in the sup metric). This follows in part from the following theorem (which we will not prove, but is handy to have around:

Theorem 7 (The Weierstrass Approximation Theorem) *The set of all polynomials defined on $[a, b]$ is dense in the set of all continuous functions on $[a, b]$*

The fact that the set of polynomials with rational coefficients is dense in the set of all polynomials then implies that the set of continuous functions is separable.

One useful characterization of separable metric spaces, which you will prove for homework, is the following:

Theorem 8 *If a metric space M is separable then there exists a countable collection of open sets $\mathcal{O}$ such that, for any open subset U of M*

$$U = \cup \{O \in \mathcal{O} | O \subseteq U\}$$

5.2 Compactness

Now we are going to move on to a really fundamental property of metric spaces: compactness. This is a property that really does guarantee our ability to find maxima of continuous functions, amongst other things. However, its definition can seem a bit odd at first glance. First, we need to define the concept of an **open cover.**

Definition 22 *Let (M,d) be a metric space, and $K \subset M$. A collection of sets $\{F_\alpha\}$ is an open cover of K if F_α is open in M for every α, and*

$$K \subset \cup_\alpha F_\alpha$$

so, quite intuitively, and open cover of a set is just a set of open sets that covers that set.

The (slightly odd) definition of a compact metric space is as follows

Definition 23 *$K \subset M$ is compact if, for every open covering $\{F_\alpha\}$ of K there exists a finite subcover - i.e. some $\{F_\alpha\}_{i=1}^n \subset \{F_\alpha\}$ such that*

$$K \subset \cup_{i=1}^n F_\alpha$$

As a first attemt to get some intuition as to what the hell is going on here, let us first think of a set that is not compact: the open interval $(0,1)$. It should be clear that the set (of sets) $F_\alpha = \{(\frac{1}{a},1)|\alpha = 1,2,...\}$ is an open cover of $(0,1)$. However, any finite subset of this open cover will not cover $(0,1)$. To see this, note that for any finite subset of F_α, there must be some a such that, for $\alpha > a$, F_a is not in the subset. But this means that $\frac{1}{a+1} \in (0,1)$ is not covered by the subcover.

Well, that tells us that (at least one) open set is not compact. Is there a more general link between compactness and whether or not a set is open? The answer is yes, but before we get to that, we want to note another important property: compact sets are bounded. To make this statement general, we have to define boundedness for general metric spaces.

Definition 24 *Let M be a metric space. A set $S \subseteq M$ is bounded if $S \subseteq B(x,\varepsilon)$ for some $x \in S$, $\varepsilon > 0$ -*

You should check that this definition of boundedness matches the definition of boundedness in $\mathbb{R}$.

Lemma 8 *Any (nonempty) compact set is bounded*

Proof. *Let S be a compact set and let $x \in S$. then for any $y \in S$*

$$y \in \cup_{i=1}^\infty B(x,i)$$

as $d(x,y) < \infty$. Thus $\{B(x,i)\}_i^\infty$ is an open cover of S. By the definition of compactness, there must be a finite subcover.$B(x,i_1), B(x,i_2), ..., B(x,i_m)$ such that

$$S \subset \cup_{j=1}^m B(x,i_j)$$

but, this implies that

$$S \subset B(x, \max\{i_1, .., i_j\})$$

so S is bounded. ■

In fact this is one of the powers of compactness: it allows us to bring the idea of finiteness into uncountable sets. The above proof relied crucially on the fact that, because of the finite subcover, we could use the max operator in a situation where otherwise we could not.

Next we will show that a compact set must also be closed.

Proposition 3 *Any compact subset of a metric space X is closed and bounded.*

Proof. *We have already proved the bounded bit of this, so all we have to show is that any compact space S is closed. If $S = X$ then there is nothing to prove, so assume not, and pick $x \in X/S$. For any $y \in S$ we can find an ε_y such that $B(x,\varepsilon_y) \cap B(y,\varepsilon_y)$ is empty. But $\cup_{y \in S} B(y,\varepsilon_y)$ is an open cover of S, so by compactness, there must be a finite subset $T \subset S$ such that $S \subset \cup_{y \in T} B(y,\varepsilon_y)$. Define $\varepsilon = \min \varepsilon_y | y \in T$, and note that $B(x,\varepsilon) \cap S = \emptyset$. Thus X/S is open and S is closed.* ■

Moreover, any closed subset of a compact metric space will also be compact

Lemma 9 *Any closed subset of a compact metric space X is compact*

Proof. *Let S be a closed subset of X, and let $\{O_\alpha\}$ be an open cover of S. Then $\{O_\alpha\} \cup [X/S]$ is an open cover of X, and therefore has a finite subcover,T. But then $T \backslash [X/S]$ is a finite subset of $\{O\}_\alpha$ that covers S* ■

Does this mean that the compactness is exactly the same as closedness and boundedness? It should come as no surprise that the answer is no - otherwise we wouldn't have gone to such pains to define the new concept. However, one metric space where the two concepts are equivalent is $\mathbb{R}^n$.

To prove this, we will first come up with a very useful alternative characterization of compactness: sequential compactness.

Theorem 9 *A subset S of a metric space X is compact if and only if every sequence in S has a subsequence that converges in S. (a property we call sequential compactness).*[4]

Before we prove this, it is worth noting how useful this is. First, it can make it easier to show that a particular space is compact, as sequential compactness is often easier to prove. Second, it means that if we know we are working in a compact metric space, we know that any sequence we are working with will have a convergent subsequence.

Proving that compactness implies sequential compactness is relatively easy.

Lemma 10 *If a subset S of a metric space X is compact then every sequence in S has a subsequence that converges in S*

Proof (Compactness implies Sequential Compactness). *Let S be a compact subset of S, and say that some sequence $\{x_n\}$ does not have a convergent subsequence. This means that the set $T = \{x_1, x_2, ...\}$ is closed (Say for some $x \notin T$, it was the case that $B(x, \varepsilon) \cap T \neq \emptyset \; \forall \; \varepsilon$, then we can construct some subsequence converging to x). Thus, as S is compact, so is T (by lemma 9). As x_n lacks a convergent subsequence, for any $m \in \mathbb{N}$ there exists $\varepsilon_m > 0$ such that $B(x_m, \varepsilon_m) = x_m$. But $\{B(x_m, \varepsilon_m)\}_{m=1}^{\infty}$is an open cover of T and as such has a finite subcover. But this implies at least one term in the sequence $\{x_n\}$ must be infinitely repeated, a contradiction.* ■

To go the other way is somewhat laboured. The strategy of the proof is as follows:

1. Show that every sequentially compact metric space is totally bounded
2. Show that for every sequentially compact space, and any open cover of that space, we can find an ε such that for every x we can find an O_x in the open cover such that $B(x, \varepsilon) \subset O_x$

In order to prove that sequential compactness implies compactness, we first need to introduce the idea of a set being totally bounded

[4] Note that the fact that X is a metric space is a crucial qualifier here. The statement is not true in more general topologies.

Definition 25 *A set S is totally bounded if, for any $\varepsilon > 0$, there exists a finite subset T of S such that $S \subset \cup_{x \in T} B(x, \varepsilon)$*

In $\mathbb{R}^n$, the boundedness and total boundness are the same, but this is not true in other spaces - total boundedness is generally a stronger concept.

What is true is that sequental compactness implies total boundedness, as we now show

Lemma 11 *Every sequentially compact space of a metric space X is totally bounded.*

Proof. *Assume not. Then these exists a set S that is sequentially compact, but for some $\varepsilon > 0$ there exists no for no finite T such that $S \subset \cup_{x \in T} B(x, \varepsilon)$. But this means we can construct a sequence with no convergent subsequence. To see this, pick $x_1 \in S$. There exists some $x_2 \in S/B(x_1, \varepsilon)$. Similarly, we must be able to find some $x_3 \in S/(B(x_1, \varepsilon) \cup B(x_1, \varepsilon))$ and so on. Constructing a sequence in such a way, we have a sequence $\{x_n\} \in S^\infty$ such that $d(x_i, x_j) > \varepsilon \ \forall \ i, \neq j$. Such a sequence can have no convergent subseqence, and so we are done* ■

So we now know that every sequentially compact space is totally bounded. We now just need one more piece

Lemma 12 *Let S be a sequentially compact subset of X and O an open cover of S. Then there exists an $\varepsilon > 0$ such that, for any $x \in S$ we can find an $O_x \in O$ such that $B(x, \varepsilon) \subset O_x$*

Proof. *Assume not, then for any $m \in \mathbb{N}$, there exists an $x^m \in S$ such that $B(x^m, \frac{1}{m})$ is not covered by any element in O. By sequential compactness, we can find a convergent subsequnce of this sequence that converges to an $x \in S$. Lets call this subsequence x_n. As O is an open cover of S we know that $x \in O^*$ for some $O^* \in O$. As O^* is open, we have $B(x, \varepsilon) \in O^*$ for some ε. But this means that, for some N, $B(x_n, \frac{\varepsilon}{2}) \subset O^*$ for all $n > N$, a contradiction.* ■

Putting these two lemma(s?) together gives us the result that we need

Proof of Theorem 2. All that remains to be shown is that any sequentially compact set is compact. Let S be a sequentially compact set and let O be an open cover of S. By the proceeding two lemmas, we can find an $\varepsilon > 0$ and a finite T such that

1. $S \subset \cup_{x \in T} B(x, \varepsilon)$

2. $B(x, \varepsilon) \subset O_x \in O$

Which is all we need. ■

To see the power of this characterization, you should check that the following immediately follow

Corollary 2 *Any closed, bounded subset of* $\mathbb{R}$ *is compact (what results do we use to get this result?)*

Corollary 3 *Any closed, bounded subset of* $\mathbb{R}^n$ *is compact*

However, more generally, compactness is a stronger concept than closedness and boundedness, as the following examples demonstrate

Example 11 *The set* $(0, 1)$ *is closed and bounded in itself but it is not compact.*

These are quite boring examples, but a more interesting exampleis the following

Example 12 *Let* $f_m : [0, 1] \to \mathbb{R}$ *be defined as* $f_m = t^m$. *Then* $F = \{f_m | m \in \mathbb{N}\} \subset \mathcal{C}[0, 1]$, *and* $cl(F)$ *is a closed and bounded subset of* $\mathcal{C}[0, 1]$, *but it is not compact.*

One final thing that we need to know about compact sets is that continuous functions map compact sets to compact sets

Theorem 10 *Let* $f : (M, d) \to (Y, \rho)$ *be a continuous function. Then if* $K \subset M$ *is compact, then so is* $f(K) \subset Y$

Proof. *Homework.* ■