Proof Techniques[1]

Math Camp 2012

Prove the following by direct proof.

1. $n(n+1)$ is an even number.

   Take any $n \in \mathbb{N}$, then n is either even or odd.

   - Suppose n is even $\Rightarrow n = 2m$ for some $m \Rightarrow n(n+1) = 2m(n+1) \Rightarrow n(n+1)$ is even.
   - Suppose n is odd $\Rightarrow n+1$ is even $\Rightarrow n+1 = 2m$ for some $m \Rightarrow n(n+1) = 2nm \Rightarrow n(n+1)$ is even. QED

2. The sum of the first n natural numbers is $\frac{1}{2}n(n+1)$.

   Let $n \in \mathbb{N}$

   - *Case 1 n is even*
     Then

$$\begin{aligned} 1+2+...+n &= (1=n)+(2+(n-1))+....+(\frac{n}{2}+(\frac{n}{2}+1)) \\ &= (n+1)+(n+1)+.....+(n+1) \\ \frac{n}{2} & \quad \textit{times} \\ &= \frac{n}{2}(n+1) = \frac{1}{2}n(n+1) \end{aligned}$$

   - *Case 1 n is odd*
     Then

$$\begin{aligned} 1+2+...+n &= (1=n)+(2+(n-1))+....+(\frac{n-1}{2}+(\frac{n+3}{2}+1))+\frac{n+1}{2} \\ &= (n+1)+(n+1)+.....+(n+1)+\frac{n+1}{2} \\ \frac{n-1}{2} & \quad \textit{times} \\ &= \frac{n-1}{2}(n+1)+\frac{n+1}{2} \\ &= \frac{n-1+1}{2}(n+1) \\ &= \frac{1}{2}n(n+1) \end{aligned}$$

3. If $6x+9y=101$, then either x or y is not an integer.

$$6x+9y=101 \Leftrightarrow 3(2x+3y)=101 \Leftrightarrow 2x+3y=\frac{101}{3} \notin \mathbb{Z}$$

   - Suppose $x \in \mathbb{Z} \Rightarrow 2x \in \mathbb{Z}$. Then it must be the case that $3y \notin \mathbb{Z}$ since the sum of two integers is also an integer and we know that $2x+3y=\frac{101}{3} \notin \mathbb{Z}$. Therefore $y \notin \mathbb{Z}$

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- Suppose $y \in \mathbb{Z} \Rightarrow 3y \in \mathbb{Z}$. Then it must be the case that $2x \notin \mathbb{Z}$ for the same argument. Therefore $x \notin \mathbb{Z}$

Therefore either x or y is not an integer

Prove the following by contrapositive.

1. $n(n+1)$ is an even number.

   We want to show that: x is odd $\Rightarrow x \neq n(n+1)$ for any $n \in \mathbb{N}$.

   Let $x \in \mathbb{N}$, x odd, then $x = 2k+1$ for some $k \in \mathbb{N}$. Suppose $x = 2k+1 = n(n+1)$ for some $n \in \mathbb{N}$

   - *Case 1 n even.* Therefore $2k+1 = 2m(2m+1)$ for some $m \in \mathbb{N}$. If so then $k = \frac{2m(2m+1)-1}{2} = m(2m+1) - \frac{1}{2} \notin \mathbb{N}$ since $m(2m+1) \in \mathbb{N}$ and $\frac{1}{2} \notin \mathbb{N}$. Therefore we have a contradiction with the fact that $k \in \mathbb{N}$
   - *Case 2 n odd.* Therefore $2k+1 = (2m-1)2m$ for some $m \in \mathbb{N}$. If so then $k = \frac{2m(2m-1)-1}{2} = m(2m-1) - \frac{1}{2} \notin \mathbb{N}$ since $m(2m-1) \in \mathbb{N}$ and $\frac{1}{2} \notin \mathbb{N}$. Therefore we have a contradiction with the fact that $k \in \mathbb{N}$

   Both cases lead to a contradiction therefore we have that $x \neq n(n+1)$ for any $n \in \mathbb{N}$.

2. If $x + y > 100$, then either $x > 50$ or $y > 50$.

   We want to show that $x < 50$ and $y < 50 \Rightarrow x + y < 100$. Let $x < 50$ and $y < 50$, then $x + y < 50 + 50 = 100$ QED

Prove the following by contradiction.

1. $n(n+1)$ is an even number.

   Suppose $n(n+1)$ is odd. Then $n(n+1) = 2k+1$ for some $k \in \mathbb{N}$, therefore $k = \frac{n(n+1)-1}{2} = \frac{n(n+1)}{2} - \frac{1}{2}$. Since either n or $n+1$ is even, the first term $\frac{n(n+1)}{2}$ is an integer, so $k \notin \mathbb{N}$. QED

2. $\sqrt{3}$ is an irrational number.

   Suppose that $\sqrt{3} \in \mathbb{Q}$, then $\sqrt{3} = \frac{p}{q}$, not both multiple of 3, then

   $$3 = \frac{p^2}{q^2} \Rightarrow p^2 = 3q^2$$

   then p^2 is multiple of 3, then p is multiple of 3, that is $p = 3k$ for some $k \in \mathbb{N}$. Therefore we have that

   $$(3k)^2 = 9k^2 = 3q^2 \Rightarrow 3k^2 = q^2$$

   then q^2 is multiple of 3, therefore q is multiple of 3. Contradiction. QED.

3. There are infinitely many prime numbers.

   Suppose that there are finitely many prime numbers, $p_1 < p_2 < ... < p_r$. Define $q = p_1 p_2 .... p_r + 1$. Suppose that p is a prime number that divides q. Then

   $$\frac{q}{p} = \frac{p_1 p_2 ... p_r + 1}{p} = \frac{p_1 p_2 ... p_r}{p} \frac{1}{p} \in \mathbb{N}$$

If $p = p_i$ for some $i \in \{1, 2, ..., r\}$ then $\frac{p_1 p_2 ... p_r}{p} \in \mathbb{N}$, but $\frac{1}{p} \notin \mathbb{N}$, therefore $\frac{q}{p} \notin \mathbb{N}$

If $p \neq p_i$ for some $i \in \{1, 2, ..., r\}$ then p is a prime number different from all the first r prime numbers. Contradiction. QED

Prove the following by induction.

1. $n(n + 1)$ is an even number.

   $P(1) : 1(1 + 1) = 2$ is even. So $P(1)$ is true.

   Then we need to prove that $P(k) \Rightarrow P(k + 1)$. Assume $k(k + 1)$ is even we need to show that $(k + 1)(k + 2)$ is even.

   If $k(k+1)$ is even, then $(k+1)(k+2) = k(k+1)+2(k+1)$ is even since $2(k+1)$ is even. QED.

2. $2n \leq 2^n$.

   $P(1) : 2 \leq 2^1 \Leftrightarrow 2 \leq 2$ true.

   Then we need to prove that $P(k) \Rightarrow P(k + 1)$. Assume $2k \leq 2^k$ we need to show that $2(k + 1) \leq 2^(k + 1)$.

   $$2^{k+1} = 2^k 2 \geq 2k2 = 2k + 2k \geq 2k + 2 = 2(k + 1)$$

   for all $k \in \mathbb{N}$ with $k > 1$

3. $\sum_{i=1}^{n} i^2 = \frac{1}{6} n (n + 1) (2n + 1)$.

   $P(1) : 1^2 = \frac{1}{6}(1 + 1)(2 + 1) = \frac{6}{6}$ then is true.

   Then we need to prove that $P(k) \Rightarrow P(k + 1)$. Assume $\sum_{i=1}^{k} i^2 = \frac{1}{6} k (k + 1) (2k + 1)$ we need to show that $\sum_{i=1}^{k+1} i^2 = \frac{1}{6}(k + 1) (k + 2) (2(k + 1) + 1)$. Then we have that

   $$\begin{aligned}
   \sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + (k+1)^2 &= \frac{1}{6} k (k + 1) (2k + 1) + (k + 1)^2 \\
   &= (k + 1) \left[\frac{1}{6} k (2k + 1) + (k + 1)\right] \\
   &= (k + 1) \left[\frac{1}{3} k^2 + \frac{7}{6} k + 1\right] \\
   &= \frac{1}{6}(k + 1) \left[2k^2 + 7k + 1\right] \\
   &= \frac{1}{6}(k + 1) [(k + 2)(2k + 3)] \\
   &= \frac{1}{6}(k + 1)((k + 1) + 1)(2(k + 1) + 1)
   \end{aligned}$$

4. The sum of the first n odd integers is n^2 (This is the first known proof by mathematical induction, attributed to Francesco Maurolico. Just in case you were interested.)

   We want to show that $\sum_{i=1}^{n}(2i - 1) = n^2$

P(1): $2-1=1^2 \Leftrightarrow 1=1$ so it's true.

Then we need to prove that $P(k) \Rightarrow P(k+1)$. Assume $\sum_{i=1}^{k}(2i-1)=k^2$ we need to show that $\sum_{i=1}^{k+1}(2i-1)=(k+1)^2$. Then we have that

$$\begin{aligned}\sum_{i=1}^{k+1}(2i-1) &= \sum_{i=1}^{k}(2i-1)+(2(k+1)-1)\\ &= k^2+2k+1=(k+1)^2\end{aligned}$$

Find the error in the following argument, supposedly by induction: If there is only one horse, then all the horses are of the same color. Now suppose that within any set of n horses, they are all of the same color. Now look at any set of $n+1$ horses. Number them $1,2,3,\ldots,n,n+1$. Consider the sets $\{1,2,3,\ldots,n\}$ and $\{2,3,4,\ldots,n+1\}$. Each set is a set of n horses, therefore they are all of the same color. But these sets overlap, therefore all horses are the same color. If n=1 then

the sets $\{1,2,3,\ldots,n\}$ and $\{2,3,4,\ldots,n+1\}$ are actually $\{1\}$ and $\{2\}$ and do not overlap, then the proof breaks down because it is not possible to do the step of $P(1) \nRightarrow P(2)$.

Prove the following (solution in Analysis solution sheet):

1. Let f and g be functions from $\mathbb{R}^k$ to $\mathbb{R}^m$ which are continuous at x. Then $h=f-g$ is continuous at x.
2. Let f and g be functions from $\mathbb{R}^k$ to $\mathbb{R}^m$ which are continuous at x. Then $h=fg$ is continuous at x.

Preference Relations

In first semester micro you will be introduced to preference relations. We say that $x \succeq y$, (read "x is weakly preferred to y") if x is at least as good as y to the agent. From this, we can derive two important relations:

- The strict preference relation, $\succ$, defined by $x \succ y \Leftrightarrow x \succeq y$ but not $y \succeq x$. The strict preference relation is read "x is strictly preferred to y".
- The indifference relation, $\sim$, defined by $x \sim y \Leftrightarrow x \succeq y$ and $y \succeq x$. The indifference relation is read "x is indifferent to y".

We say that a preference relation is rational if:

- $\forall\ x, y$, either $x \succeq y$ or $y \succeq x$.
- $\forall\ x, y, z$, if $x \succeq y$ and $y \succeq z$, then $x \succeq z$.

Prove the following two statements given that preferences are rational:

1. If $x \succ y$ and $y \succ z$, then $x \succ z$.

   Suppose that $x \succ y$ (1) and $y \succ z$ (2), then we have that (1) means that $x \succ y \Leftrightarrow x \succeq y$ (a) but not $y \succeq x$ (b); while (2) means that $y \succ z \Leftrightarrow y \succeq z$ (c) but not $z \succeq y$ (d). So from (a) and (c) we have that, given that preferences are rational (transitivity) it is true that $x \succeq z$.

   Now we need to prove that is not the case that $z \succeq x$. Let assume, by contradiction that it is the case that $z \succeq x$. From (c) we know that $y \succeq z$ so by transitivity again we have that $y \succeq x$ which contradicts statement (b). QED.

2. If $x \sim y$ and $y \sim z$, then $x \sim z$.

$$x \sim y \Leftrightarrow x \succeq y[a] \& y \succeq x[b]$$
$$y \sim z \Leftrightarrow y \succeq z[c] \& z \succeq y[d]$$

From [a] and [c], using rationality we have that $x \succeq z$ and from [b] and [d] we have that $z \succeq x$, therefore we have that $x \sim z$. QED