Basic Proof Techniques

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1 Basic Notation

The following is standard notation for proofs:

• \( A \Rightarrow B \). \( A \) implies \( B \).

• \( A \Leftarrow B \). \( B \) implies \( A \).

Note that \( A \Rightarrow B \) does not mean \( B \Rightarrow A \). Example: If \( A \) a person eats two hot dogs, she also \( B \) eats one hot dog. However, if \( B \) a person eats one hot dog, that does not imply that she also \( A \) eats two hot dogs.

• \( A \iff B \). \( A \) implies \( B \) and \( B \) implies \( A \).

Another way of saying this is that \( A \) holds if and only if (iff) \( B \) holds, or that \( A \) is equivalent to \( B \).

• \( \neg A \). Not \( A \), or the negation of \( A \).

Example: If \( A \) is the event that \( x \leq 10 \), then \( \neg A \) is the event that \( x > 10 \).

It is common to use mathematical symbols for words while writing proofs in order to write faster. The following are commonly used symbols:

\( \forall \) For all, for any

\( \exists \) There exists

\( \in \) Is contained in, is an element of

\( \ni \) Such that, contains as an element

\( \subseteq \) Is a subset of

QED Latin for “quod erat demonstrandum”, or “which was to be proven”. A common way to signal to the reader that you have successfully concluded your proof.

2 Proofs

We seek for ways to prove that \( A \Rightarrow B \).
2.1 Direct Proofs

2.1.1 Deductive Reasoning

A direct proof by deductive reasoning is a sequence of accepted axioms or theorems such that \( A_0 \Rightarrow A_1 \Rightarrow A_2 \Rightarrow \cdots \Rightarrow A_{n-1} \Rightarrow A_n \), where \( A = A_0 \) and \( B = A_n \). The difficulty is finding a sequence of theorems or axioms to fill the gaps.

**Example:** Prove the number three is an odd number.

Proof: A number \( q \) is odd if there exists an integer \( m \) such that \( q = 2m + 1 \). Let \( m = 1 \). Then \( 2m + 1 = 3 \). Therefore three is an odd number. QED

2.1.2 Contrapositive

A contrapositive proof is just a direct proof of the negation. It makes use of the fact that the statement \( A \Rightarrow B \) is equivalent to the statement \( \neg B \Rightarrow \neg A \). For example, if \( (A) \) all people with driver’s licenses are \( (B) \) at least 16 years old, then if you are not \( (\neg B) \) 16 years old, then you do not \( (\neg A) \) have a driver’s license. So proving \( A \Rightarrow B \) is really the same as proving \( \neg B \Rightarrow \neg A \).

**Example:** Let \( x \) and \( y \) be two positive numbers. Prove that if \( xy > 9 \), then \( x > 3 \) or \( y > 3 \).

Proof: Suppose that both \( x \leq 3 \) and \( y \leq 3 \). Then \( xy \leq 9 \). QED (Here \( A: xy > 9 \), \( B: x > 3 \) or \( y > 3 \). In order to prove \( A \Rightarrow B \) we proved \( \neg B \Rightarrow \neg A \).)

2.2 Indirect Proofs

2.2.1 Contradiction

Suppose that we are trying to prove a proposition \( A \), and we cannot prove it directly. However, we can show that all other alternatives to \( A \) are impossible. Then we have indirectly proved that \( A \) must be true. Therefore, the we can prove \( A \Rightarrow B \) by first assuming that \( A \not= B \) and finding a contradiction. In other words, we start off by assuming that \( A \) is true but \( B \) is not. If this leads to a contradiction, then either \( B \) was actually true all along, or \( A \) was actually false. But since we assume \( A \) is true, then it must be that \( B \) is true, and we have a proof by contradiction.

**Example:** Prove that \( \sqrt{2} \) is an irrational number.

Proof: Suppose not. Then \( \sqrt{2} \) is a rational number, so it can be expressed in the form \( \frac{p}{q} \), where \( p \) and \( q \) are integers which are not both even. This implies that

\[
2 = \frac{p^2}{q^2} \Rightarrow 2q^2 = p^2,
\]

which implies that \( p^2 \) is even, which in turn implies that \( q^2 \) is not even. The fact that \( p^2 \) is even also implies that \( p \) is even, so there exists a integer \( m \) such that \( 2m = p \). This implies

\[
4m^2 = p^2 = 2q^2 \Rightarrow q^2 = 2m^2,
\]

which means that \( q \) is even, a contradiction. QED

2.2.2 Induction

Induction can only be used for propositions about integers or indexed by integers. Consider a list of statements indexed by the integers. Call the first statement \( P(1) \), the second \( P(2) \), and the \( n \)th
statement $P(n)$. If we can prove the following two statements about the sequence, then every statement in the entire sequence must be true:

1. $P(1)$ is true.
2. If $P(k)$ is true, then $P(k + 1)$ is true.

Induction works because by 1., $P(1)$ is true. By 2., $P(2)$ is true since $P(1)$ is true. Then $P(3)$ is true by 2. again, and so is $P(4)$ and $P(5)$ and $P(6)$, until we show that all the $P$'s are true. Notice that the number of propositions need not be finite.

**Example:** Prove that the sum of the first $n$ natural numbers is $\frac{1}{2}n(n + 1)$.

Proof: Let $n = 1$. Then $\frac{1}{2} \cdot 1(1 + 1) = \sum_{j=1}^{1} j = 1$. Now let $n = k$, and assume that $\sum_{j=1}^{k} j = \frac{1}{2}k(k + 1)$.

We add $k + 1$ to both sides to get

$$\sum_{j=1}^{k+1} j = \frac{1}{2}k(k + 1) + k + 1 = \left(\frac{1}{2}k + 1\right)(k + 1) = \frac{1}{2}(k + 1)((k + 1) + 1).$$

QED

### 2.3 Epsilon-Delta Arguments

A lot of definitions and proofs in real analysis use the "$\epsilon$ and $\delta$" concept. For example, recall the definition of the limit of a function: We write $\lim_{x \to p} f(x) = q$ if for every $\epsilon > 0$ we can find $\delta > 0$ such that $|f(x) - q| < \epsilon$ for all $x$ for which $|x - p| < \delta$.

It is important to get the quantifiers correct: *For every $\epsilon$ there exists $\delta$* such that... This means that $\delta$ will change with $\epsilon$ - for some value $\epsilon_1$ we'll be able to find $\delta_1$ so that the statement holds, and for some other value $\epsilon_2$ we'll find $\delta_2$ which may differ from $\delta_1$. The opposite would be *there exists $\delta$ such that for every $\epsilon$ it holds that...*. Here there is only one $\delta$ which has to fit all different values of $\epsilon$.

When we prove a statement involving an $\epsilon$-$\delta$ definition, we start by ascribing a fixed, but unknown, value to $\epsilon$ ("fix $\epsilon > 0$"). Then we try to find value of $\delta$ that makes the statement in question come true. This $\delta$ will usually be a function of $\epsilon$. This completes the proof since $\epsilon$ could have been anything: For every $\epsilon$ we have found a $\delta$ such that the statement holds.

When we want to show that a certain $\epsilon$-$\delta$ statement does not hold, we usually choose one particular $\epsilon$ for which the statement should not be true ("Let $\epsilon = 0.5$"). Then we proceed by contradiction: We pretend there exists some $\delta$ that fits our $\epsilon$ and show that this leads to a contradiction. Ergo, no $\delta$ can fit our particular $\epsilon$. Therefore it is not true that for all $\epsilon$ we can find a fitting $\delta$.

**Example 1** Prove that $\lim_{x \to 0} x^2 + 1 = 1$.

Proof: Fix $\epsilon > 0$. How small does $\delta$ have to be so that $|(x^2 + 1) - 1| = |x^2| < \epsilon$ for all $x$ for which $|x| < \delta$? $\delta = \sqrt{\epsilon}$ works: If $|x| < \sqrt{\epsilon}$ then $|x|^2 < (\sqrt{\epsilon})^2$. $|x|^2 = |x^2|$ and $(\sqrt{\epsilon})^2 = \epsilon$, therefore $|x^2| < \epsilon$.

QED.

**Example 2** Show that $f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$ is not continuous.

Recall that $f$ is continuous at a point $p$ if for every $\epsilon > 0$ we can find $\delta > 0$ such that $|f(x) - f(p)| < \epsilon$ for all $x$ for which $|x - p| < \delta$. $f$ is probably not continuous is at 0. We therefore take $p = 0$ and show that the above definition of continuity does not hold. It suffices to show that for one particular $\epsilon$ we
cannot find a fitting $\delta$. We prove this by contradiction.

Proof: Let $\epsilon = 1$. Suppose there exists $\delta$ such that $|f(x) - f(0)| < \epsilon$ for all $x$ for which $|x| < \delta$. Plugging in, this means $|f(x) - 1| < 1$ for all $x$ for which $|x| < \delta$. Set $x = -\frac{\delta}{2}$. Then $|x| < \delta$ but $|f(x) - 1| = |-1 - 1| = 2$ which is not $< 1$. This contradicts $|f(x) - 1| < 1$ for all $x$ for which $|x| < \delta$. Therefore there exists not $\delta$ such that the definition of continuity becomes true. $f$ is not continuous at 0 and therefore not a continuous function. QED.

Exercises

1. Negate the definition of convergence for a sequence. (The definition is in Real Analysis part.)

2. Negate the definition of continuity.

3. Let $f(x) = \begin{cases} 1 & \text{if } x \neq 0 \\ 1.1 & \text{if } x = 0 \end{cases}$. Prove that $f$ is not continuous at 0.

4. Prove that $\lim_{x \to 0} \frac{1}{x} \neq K$ for any number $K$.

5. Let $f$ and $g$ be two continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Show that $f + g$ is continuous.
3 Homework

Prove the following by direct proof.

1. \( n(n + 1) \) is an even number.
2. The sum of the first \( n \) natural numbers is \( \frac{1}{2}n(n + 1) \).
3. If \( 6x + 9y = 101 \), then either \( x \) or \( y \) is not an integer.

Prove the following by contrapositive.

1. \( n(n + 1) \) is an even number.
2. If \( x + y > 100 \), then either \( x > 50 \) or \( y > 50 \).

Prove the following by contradiction.

1. \( n(n + 1) \) is an even number.
2. \( \sqrt{3} \) is an irrational number.
3. There are infinitely many prime numbers.

Prove the following by induction.

1. \( n(n + 1) \) is an even number.
2. \( 2n \leq 2^n \).
3. \( \sum_{i=1}^{n} i^2 = \frac{1}{6}n(n + 1)(2n + 1) \).
4. The sum of the first \( n \) odd integers is \( n^2 \) (This is the first known proof by mathematical induction, attributed to Francesco Maurolico. Just in case you were interested.)

Find the error in the following argument, supposedly by induction:

If there is only one horse, then all the horses are of the same color. Now suppose that within any set of \( n \) horses, they are all of the same color. Now look at any set of \( n + 1 \) horses. Number them \( 1, 2, 3, \ldots, n, n + 1 \). Consider the sets \( \{1, 2, 3, \ldots, n\} \) and \( \{2, 3, 4, \ldots, n + 1\} \). Each set is a set of \( n \) horses, therefore they are all of the same color. But these sets overlap, therefore all horses are the same color.

Prove the following (solution in Analysis solution sheet):

1. Let \( f \) and \( g \) be functions from \( \mathbb{R}^k \) to \( \mathbb{R}^m \) which are continuous at \( x \). Then \( h = f - g \) is continuous at \( x \).
2. Let \( f \) and \( g \) be functions from \( \mathbb{R}^k \) to \( \mathbb{R}^m \) which are continuous at \( x \). Then \( h = fg \) is continuous at \( x \).

In first semester micro you will be introduced to preference relations. We say that \( x \succeq y \) (read “\( x \) is weakly preferred to \( y \)”) if \( x \) is at least as good as \( y \) to the agent. From this, we can derive two important relations:
• The strict preference relation, $\succ$, defined by $x \succ y \iff x \succeq y$ but not $y \succeq x$. The strict preference relation is read “$x$ is strictly preferred to $y$”.

• The indifference relation, $\sim$, defined by $x \sim y \iff x \succeq y$ and $y \succeq x$. The indifference relation is read “$x$ is indifferent to $y$”.

We say that a preference relation is rational if:

• $\forall x, y$, either $x \succeq y$ or $y \succeq x$.

• $\forall x, y, z$, if $x \succeq y$ and $y \succeq z$, then $x \succeq z$.

Prove the following two statements given that preferences are rational:

1. If $x \succ y$ and $y \succ z$, then $x \succ z$.

2. If $x \sim y$ and $y \sim z$, then $x \sim z$. 