Introduction to Real Analysis

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1 Sets

Sets are the basic objects of mathematics. In fact, they are so basic that there is no simple and precise definition of what a set actually is. For our purposes it suffices to think of a set as a collection of objects.

Sets are denoted as a list of their elements: $A = \{a, b, c\}$ means the set A consists of the elements a, b, c. The number of elements in a set can be finite or infinite. For example, the set of all even integers $\{2, 4, 6, \ldots\}$ is an infinite set. We use the notation $a \in A$ to say that a is an element of the set A. Suppose we are given a set X. A is a *subset* of X if all elements in A are also contained in X: $a \in A \Rightarrow a \in X$. It is denoted $A \subset X$. The *empty set* is the set that contains no elements. It is denoted $\{\}$ or $\varnothing$. Note that any statement about the elements of the empty set is true - since there are no elements in the empty set.

1.1 Set Operations

The *union* of two sets A and B is the set consisting of the elements that are in A or in B (or in both). It is denoted $A \cup B$. For example, if $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$ then $A \cup B = \{1, 2, 3, 4\}$. The *intersection* of two sets A and B is the set consisting of the elements that are in A and in B. It is denoted $A \cap B$. For example, if $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$ then $A \cap B = \{2, 3\}$. Suppose A is a subset of X. Then the *complement* of A in X, denoted A^C is the set of all elements in X that are not contained in A: $A^C = \{x \in X \text{ such that } x \notin A\}$. The *Cartesian Product* of two sets A and B, denoted $A \times B$, is the set of all possible ordered pairs whose first component is an element of A and whose second component is an element of B: $A \times B = \{(a, b) \text{ such that } a \in A \text{ and } b \in B\}$.

1.2 Finite, Countable, and Uncountable Sets

A set A is set to be *finite* if there exists a bijective ("one-to-one and onto") function f mapping from a set $1, 2, \ldots, n$ to A. This simply means the set has a finite number of elements - to each element we can assign exactly one number from $1, \ldots, n$. A set A is *infinite* if it is not finite. A set A is *countable* if there exists a bijective function f mapping from the set of positive integers to A. This means the set as infinitely many elements but that theoretically we could count them all (if we had infinite time to do so). A set A is *uncountable* if it is neither finite nor countable. The most prominent uncountable set is the set of real numbers $\mathbb{R}$. The set of rational numbers $\mathbb{Q}$ and the set of integers $\mathbb{Z}$ are both countable.

2 Metric Spaces

Sets form the basis of mathematics, but we cannot do much with them in economics unless we define an additional structure on them - the notion of distance between two elements. If we can measure the distance between elements in a set, the set is called a metric space. The elements of a metric space are called points.

Definition A set X is said to be a *metric space* if with any two points p and q of X there is associated a real number $d(p,q)$ called the *distance* from p to q such that

1. $d(p,q) > 0$ if $p \neq q$, and $d(p,q) = 0$ if $p = q$;
2. $d(p,q) = d(q,p)$;
3. $d(p,q) \leq d(p,r) + d(r,q)$ for any $r \in X$

Any function with these three properties is called a *distance function* or *metric*.

The most important metric spaces we encounter in economics are the Euclidean spaces $\mathbb{R}^n$, in particular the real numbers $\mathbb{R}$ and the real plane $\mathbb{R}^2$. In these spaces, the most commonly used distance is the Euclidean distance, which is defined as

$$d(x,y) = ||x-y|| = \sqrt{\sum_{i=1}^{N}(x_i - y_i)^2}.$$

Other distances could be street or grid distances. For example, if you are on the south-west corner of a city block and you want to go to the north-east corner of the same block, you must travel east one block and north one block. The grid distance you walked is two blocks, whereas the Euclidean distance is $\sqrt{2}$ blocks. The grid distance can be defined mathematically as

$$d(x,y) = \sum_{i=1}^{N} |x_i - y_i|.$$

2.1 Some Definitions

Equipped with a distance d we can define the following subsets of a metric space X (you can simply think of d as the Euclidean distance and of X as $\mathbb{R}^N$):

Open and Closed Balls The set $B(x,r) = \{y \in X : \ d(x,y) < r\}$ is called the open ball $B(x,r)$ with center x and radius r. The set $B(x,r) = \{y \in X : \ d(x,y) \leq r\}$ is called the closed ball $B(x,r)$ with center x and radius r.In contrast to an open ball, a closed ball contains the points of the boundary where $d(x,y) = r$. Sometimes the radius is labeled ϵ instead of r and then the ball is also called epsilon ball. Note that in $\mathbb{R}$ an open ball is simply an open interval $(x-r, x+r)$, i.e. the set $\{y \in \mathbb{R} : x - r < y < x + r\}$, and a closed ball is simply a closed interval $(x-r, x+r)$, i.e. the set $\{y \in \mathbb{R} : x - r \leq y \leq x + r\}$.

Open and Closed Sets A set $U \subset X$ is *open* if $\forall\ x \in U$ there exists $r > 0$ such that $B(x,r) \subset U$. In English: A set is open if for any point x in the set we can find a small ball around x that is also contained in the set. Basically an open set is a set that does not contain its boundary since any ball around a point on the boundary will be partially in the set and partially out of the set. For example, the interval $(0, 1)$ is open in $\mathbb{R}$ since for any point x in $(0, 1)$, we can find a small interval around x that is also contained in $(0, 1)$.

A set $U \subset X$ is *closed* if its compliment is open. An equivalent definition is that a set is closed iff $\forall$ sequences $\{x_k\}$ with $x_k \in U \ \forall\ k$ and $\{x_k\} \to x$, then $x \in U$. Basically a closed set is a set that contains its boundary (since the complement of that set does not contain the boundary and is thus open). The definition using sequences says that if a sequence $\{x_k\}$ gets arbitrarily close to a point x while staying in the closed set then the point x also has to be in the set. For example, the interval $[0, 1]$ is closed in $\mathbb{R}$ since its complement, the set $(-\infty, 0) \cup (1, \infty)$, is open. Note that a set can be open (e.g. $(0, 1)$), closed (e.g. $[0, 1]$), neither (e.g. $(0, 1]$) or both ($\{\}$, $\mathbb{R}$)!

Problem Show A set $U \subset X$ is closed if and only if $\forall$ sequences $\{x_k\}$ with $x_k \in U \ \forall\ k$ and $\{x_k\} \to x$, then $x \in U$.

Bounded Set A set $U \subset X$ is *bounded* if $\exists\ r > 0$ and $x \in X$ such that $U \subset B(x, r)$.
This is an easy one: A set is bounded if we can fit it into a large enough ball around some point. A set is not bounded if no matter how large we choose the radius of the ball, the set will not be completely contained in it.

The next two definitions concern the Euclidean space $\mathbb{R}^N$ only.

Compact Set A set $U \subset \mathbb{R}^N$ is *compact* if it is closed and bounded. So we can think of a compact set in $\mathbb{R}^N$ as a set that fits into a ball and contains its boundary. In a general metric space, the definition of compact set is different, but we do not have to deal with it here.

Convex Combination & Convex Set Given any finite number of points $\{x_1, \ldots x_n\}$, $x_i \in \mathbb{R}^N$, a point $z \in \mathbb{R}^N$ is a *convex combination* of the points $\{x_1, \ldots x_n\}$ if $\exists\ \lambda \in \mathbb{R}^N_+$ satisfying $\sum_{i=1}^N \lambda_i = 1$ such that $z = \sum_{i=1}^N \lambda_i x_i$. For example, the convex combinations of two points in $\mathbb{R}^2$ form the line segment connecting the two points.

A set is *convex* if the convex combination of any two points in the set is also contained in the set. Another way of stating this is if $x_1, x_2 \in X$, then $\alpha x_1 + (1 - \alpha)x_2 \in X$, where $\alpha \in [0, 1]$. The second definition says that a set is convex if you can draw a straight line between any two points in the set that is completely contained in the set. For example, if you pick any two points in the unit disk, the line connecting them is also contained in the unit disk. On the unit circle that is not the case: $(1, 0)$ and $(-1, 0)$ are both on the unit circle, but the line connecting them goes through $(0, 0)$, which is not on the unit circle. The unit disc is a convex set in $\mathbb{R}^2$, while the unit circle is not.

Problem (Jensen's Inequality) Let $f : \mathbb{R}^N \to \mathbb{R}$. We define f to be a *convex* function if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y), \quad 0 \leq \lambda \leq 1$$

for every $x, y \in \mathbb{R}^N$. Show that f is convex if and only if

$$f\left(\sum_{i=1}^N \lambda_i x_i\right) \leq \sum_{i=1}^N \lambda_i f(x_i)$$

whenever $\lambda_i \geq 0$ for all i and $\sum_{i=1}^N \lambda_i = 1$.

Some Facts ...to be proved as exercises.

1. Open balls are open sets.
2. Any union of open sets is open.
3. The finite intersection of open sets is open.
4. Any intersection of closed sets is closed.
5. The finite union of closed sets is closed.

3 Sequences

Definition A *sequence* is an assignment of the elements in some set to the natural numbers. A sequence is denoted as a set with elements labeled from zero (or one) to a finite number or infinity:

$$\text{Finite sequence: } \{x_n\}_{n=0}^{N} = \{x_1, x_2, \ldots, x_N\}$$

$$\text{Infinite sequence: } \{x_n\}_{n=0}^{\infty} = \{x_1, x_2, \ldots\}$$

Examples:

$$\{x_n\}_{n=0}^{\infty} = \{1, 1, 2, 3, 5, 8, \ldots\}$$
$$\{x_n\}_{n=0}^{\infty} = \{1, 0, 1, 0, \ldots\}$$
$$\{x_n\}_{n=0}^{\infty} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$$

Sequences can also be defined as functions of n.

Examples:

$$a(n) = n^2 \Rightarrow \{x_n\}_{n=0}^{\infty} = \{0, 1, 4, 16, \ldots\}$$
$$b(n) = \frac{n}{n+1} \Rightarrow \{x_n\}_{n=0}^{\infty} = \left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\right\}$$
$$c(n) = \sqrt{\ln(n)} \Rightarrow \{x_n\}_{n=1}^{\infty} = \left\{\sqrt{\ln(1)}, \sqrt{\ln(2)}, \ldots\right\}$$
$$d(n) = \frac{e^n}{n} \Rightarrow \{x_n\}_{n=1}^{\infty} = \left\{e, \frac{e^2}{2}, \frac{e^3}{3}, \ldots\right\}$$

Subsequences Given a sequence $\{x_n\}$, consider the sequence of positive integers $\{n_k\}$ such that $n_1 < n_2 < n_3 < \ldots$. Then the sequence $\{x_{n_k}\}$ is called a subsequence of $\{x_n\}$.

Example: Let $\{x_n\} = \{\frac{1}{n}\}$ and $\{n_k\}$ be the sequence of prime numbers. Then $\{x_{n_k}\} = \{1, \frac{1}{3}, \frac{1}{5}, \ldots\}$.

3.1 Properties of Sequences in $\mathbb{R}$

We first look at sequences in one dimension, i.e. sequences in $\mathbb{R}$. The properties and definitions generalize easily to sequences of higher dimensions, i.e. sequences in $\mathbb{R}^N$, since a sequence in $\mathbb{R}^N$ can be essentially viewed as a vector of N sequences in $\mathbb{R}$. All of these definitions also apply to general metric spaces.

Boundedness A sequence $\{x_n\}_{n=0}^{\infty}$ in $\mathbb{R}$ is *bounded above* if there exists a number M_a such that all elements of the sequence are less than M_a: $x_n \leq M_a \; \forall \, n \in \mathbb{N}$. We call M_a an *upper bound*. A sequence $\{x_n\}_{n=0}^{\infty}$ in $\mathbb{R}$ is *bounded below* if there exists a number M_b such that all elements of the sequence are greater than M_b: $x_n \geq M_b \; \forall \, n \in \mathbb{N}$. We call M_b a lower bound. A sequence in $\mathbb{R}$ is *bounded* if it is bounded both above and below. The smallest number M_a which is an upper bound of a sequence is called the *least upper bound* or *supremum*, while the largest number M_b which is a lower bound of the sequence is called the *greatest lower bound* or *infimum*.

Example: The sequence

$$\{x_n\}_{n=1}^{\infty} = \{0, 1, 0, 2, 0, 3, 0, 4, \dots\}$$

has zero as its greatest lower bound and has no upper bound. Therefore, the sequence is bounded below, but is not bounded since it is not bounded above.

Increasing and Decreasing A sequence is

1. Monotonically increasing or non-decreasing iff $x_n \leq x_{n+1}$.
2. Strictly monotonically increasing iff $x_n < x_{n+1}$.
3. Monotonically decreasing or non-increasing iff $x_n \geq x_{n+1}$.
4. Strictly monotonically decreasing iff $x_n > x_{n+1}$.
5. Monotone if it is either monotonically increasing or decreasing.

3.2 Limits, Convergence, and Divergence

Definition A sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathbb{R}$ has a *limit* $L \in \mathbb{R}$ iff for each $\epsilon > 0$, $\exists \; K \in \mathbb{Z}^{++}$ such that if $n \geq K$, then $|x_n - L| < \epsilon$. We write $\{x_n\}_{n=1}^{\infty} \to L$ or $\lim_{n\to\infty} x_n = L$.
In other words, there must be a number K such that all elements after the Kth element must be in the epsilon ball $B_\epsilon(L)$. We say a sequence *converges* if it has a limit. If it has no limit, then we say it *diverges*.

Example: Prove the sequence $\{x_n\}_{i=1}^{\infty} = \left\{1, \frac{1}{2}, \frac{1}{3}, \dots\right\}$ converges.

Proof: It suffices to show the sequence has a limit. Consider $L = 0$. We must show that for each $\epsilon > 0$, $\exists \; K \in \mathbb{Z}^{++}$ such that if $n \geq K$, then $|x_n| < \epsilon$.

$$|x_n| < \epsilon \Rightarrow |\frac{1}{n}| < \epsilon \Rightarrow \frac{1}{n} < \epsilon \Rightarrow 1 < n\epsilon \Rightarrow n > \frac{1}{\epsilon}.$$

Choose $\epsilon > 0$. Then any integer $K > \frac{1}{\epsilon}$ works. QED

To see that $K > \frac{1}{\epsilon}$ actually works, just let ϵ different numbers. For example, let $\epsilon = 1$. Then our chosen K must be greater than 1. From our sequence, we can easily see that any number $\frac{1}{n}$ where $n > 1$ is going to be less than $\epsilon = 1$. If $\epsilon = \frac{1}{2}$, then our K must be greater than 2. We can see that for every $n > 2$, $\frac{1}{n} < \epsilon = \frac{1}{2}$.

Some Facts ...to be proved as exercises.

1. A sequence can only have at most one limit.
2. If $\{x_n\}_{n=1}^{\infty} \to x$, and $\{y_n\}_{n=1}^{\infty} \to y$, then $\{x_n + y_n\}_{n=1}^{\infty} \to x + y$.

3. If $\{x_n\}_{n=1}^{\infty} \to x$, and $\{y_n\}_{n=1}^{\infty} \to y$, then $\{x_n y_n\}_{n=1}^{\infty} \to xy$.
4. If $\{x_n\}_{n=1}^{\infty} \to x$, $x_n \neq 0$ for any n, and $x \neq 0$, then $\{\frac{1}{x_n}\}_{n=1}^{\infty} \to \frac{1}{x}$.
5. Monotone Convergence Theorem: Let $\{x_n\}$ be a bounded and monotone sequence in $\mathbb{R}$. Then $\{x_n\}$ converges. (Proof not on the solution sheet but on Wikipedia.)
6. Bolzano-Weierstrass Theorem: Every bounded sequence in $\mathbb{R}$ has a convergent subsequence. (Hint: Show that every bounded sequence has a monotone subsequence. Then apply the monotone convergence theorem.)

3.3 Sequences in $\mathbb{R}^N$

A sequence $\{x_n\}_{n=0}^{\infty}$ in $\mathbb{R}^N$ can be essentially viewed as a vector of N sequences in $\mathbb{R}$:

$$\{x_n\}_{n=0}^{\infty} = \begin{pmatrix} \{(x_1)_n\}_{n=0}^{\infty} \\ \vdots \\ \{(x_N)_n\}_{n=0}^{\infty} \end{pmatrix}$$

Let's see how some of the properties above extend to multidimensional sequences: A sequence $\{x_n\}_{n=0}^{\infty}$ in $\mathbb{R}^N$ is bounded if there exists a number M such that all elements of the sequence are less than distance M from the origin, i.e. $d(x_n, 0) \leq M \forall n \in \mathbb{N}$. This is equivalent to the statement that each of the component sequences $\{(x_1)_n\}_{n=0}^{\infty}, \ldots, \{(x_N)_n\}_{n=0}^{\infty}$ is bounded. A sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathbb{R}^N$ has a limit $L \in \mathbb{R}^N$ iff for each $\epsilon > 0$, $\exists\ K \in \mathbb{Z}^{++}$ such that if $n \geq K$, then $d(x_n, L) < \epsilon$. In other words, there must be a number K such that all elements after the Kth element must be in the epsilon ball $B_\epsilon(L)$.

Theorem A sequence of vectors in $\mathbb{R}^N$ converges if and only if all the component sequences converge in $\mathbb{R}$.

Proof: Exercise.

Example Consider the sequence $\{x_n\} = \{(\frac{1}{n}, 0)\}$ in $\mathbb{R}^2$. Then $\{x_n\} \to (0, 0)$.

4 Continuity

Here we revisit the definitions of limit of a function and continuity. In the Single Variable Calculus notes, we considered functions mapping from $\mathbb{R}$ to $\mathbb{R}$. Here we generalize the definitions to functions mapping from one metric space to another. The only difference is that instead of measuring distances by absolute values we now use a general metric d.
In the following, let X and Y be two metric spaces endowed with metrics d_X and d_Y respectively, and let f be a function from X to Y.

Limit We write $f(x) \to q$ as $x \to p$ or

$$\lim_{x \to p} f(x) = q$$

if for every $\epsilon > 0$ we can find $\delta > 0$ such that $d_Y(f(x), q) < \epsilon$ for all x for which $d_X(x, p) < \delta$. q is called the limit of f at the point p.

Continuity The function f is called continuous at the point $p \in X$ if

$$\lim_{x \to p} f(x) = f(p).$$

This is equivalent to saying that for every $\epsilon > 0$ we can find $\delta > 0$ such that $d_Y(f(x), f(p)) < \epsilon$ for all x for which $d_X(x,p) < \delta$. f is called continuous if it is continuous at every point of X.

Two Theorems

1. Let X be a compact subset of $\mathbb{R}$ and let f be a continuous function from $\mathbb{R}$ to $\mathbb{R}$. Then $f(X)$ is compact. (You'll need more than we cover here to prove this, so don't try unless you know what you're doing.)

2. Let X be a compact subset of $\mathbb{R}$ and let f be a continuous function from $\mathbb{R}$ to $\mathbb{R}$. Let M_a be the least upper bound of $f(X)$ and let M_b be the greatest lower bound of $f(X)$. Then there are points p, q in X such that $f(p) = M_a$ and $f(q) = M_b$. This is the Weierstrass Theorem we'll use in optimization.
   Proof: $f(X)$ is compact by theorem 1, therefore closed and bounded. By definition M_a lies on the boundary of $f(X)$. Since closed sets contains their boundary points, M_a has to actually be in $f(X)$. That means there exists some point p in X such that $f(p) = M_a$. Analogously for M_b. QED.

5 Homework

State whether the following sets are open, closed, neither, or both:

1. $\{(x, y) : -1 < x < 1, y = 0\}$
2. $\{(x, y) : x, y \text{ are integers}\}$
3. $\{(x, y) : x + y = 1\}$
4. $\{(x, y) : x + y < 1\}$
5. $\{(x, y) : x = 0 \text{ or } y = 0\}$

Prove the following:

1. Open balls are open sets
2. Any union of open sets is open
3. The finite intersection of open sets is open
4. Any intersection of closed sets is closed
5. The finite union of closed sets is closed
6. Let f and g be functions from $\mathbb{R}^k$ to $\mathbb{R}^m$ which are continuous at x. Then $h = f - g$ is continuous at x.
7. Let f and g be functions from $\mathbb{R}^k$ to $\mathbb{R}^m$ which are continuous at x. Then $h = fg$ is continuous at x.

Find the greatest lower bound and the least upper bound of the following sequences. Also, prove whether they are convergent or divergent:

1. $\{x_n\}_{i=1}^{\infty} = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\right\}$
2. $\{x_n\}_{i=1}^{\infty} = \{-1, 1, -1, 1, \ldots\}$
3. $\{x_n\}_{i=1}^{\infty} = \left\{-\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, -\frac{5}{6}, \ldots\right\}$

Prove the following:

1. A sequence can only have at most one limit.
2. If $\{x_n\}_{n=1}^{\infty} \to x$ and $\{y_n\}_{n=1}^{\infty} \to y$, then $\{x_n + y_n\}_{n=1}^{\infty} = x + y$.
3. A sequence of vectors in $\mathbb{R}^N$ converges iff all the component sequences converge in $\mathbb{R}$.
4. The sequence $\{x_n\}_{n=1}^{\infty} = \left\{(1, \frac{1}{2}), (1, \frac{1}{3}), (1, \frac{1}{4}), \ldots\right\}$ converges to $(1, 0)$.
5. The sequence $\{x_n\}_{n=1}^{\infty} = \left\{(\frac{1}{2}, \frac{1}{2}), (\frac{2}{3}, \frac{1}{3}), (\frac{3}{4}, \frac{1}{4}), \ldots\right\}$ converges to $(1, 0)$.