# Single Variable Calculus

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## 1 Some Notation

| | |
|---|---|
| $\{a, b, c\}$ | a set containing the elements a, b, c |
| $\forall$ | for all, for any |
| $\exists$ | there exists |
| $\in$ | ss contained in, is an element of |
| $\ni$ | contains as an element |
| $\subset$ | is a subset of |
| $[a, b]$ | the closed interval from $a$ to $b$, the set of points $x$ for which $a \leq x \leq b$ |
| $(a, b)$ | the open interval from $a$ to $b$, the set of points $x$ for which $a < x < b$ |

## 2 Functions

Functions are the main mathematical objects that are used in Economics. A function takes an object (usually a number or an array of numbers) and assigns exactly one other object (usually another number or another array of numbers) to it. For example, a utility function takes a consumption bundle and assigns it a utility level. A production function takes an array of inputs and assigns it an array of outputs.

**Definition** A *function* $f$ from a set $A$ to a set $B$ is a rule that assigns to each element of $A$ one and only one object in $B$. It is denoted $f : A \to B$. The set $A$ is called the *domain* of $f$ and the set $B$ is called the *target space*.

Note that a function always assigns *exactly one* element of set B to an element of set A. If it assigns more than one element of B to an element of A, it is not a function but rather a correspondence. A function can, however, assign the same element of set B to two different elements of set A. Example: $f : x \mapsto x^2$ assigns the number 4 to both 2 and $-2$.

Here are some more important concepts associated with functions:

**Image, Range, and Preimage** Let $A$ and $B$ be two sets and let $f$ be a function from $A$ to $B$. For a set $C \subset A$, $f(C)$ is the set of all elements $f(x)$ for $x \in C$, i.e. $f(C) = \{b \in B : b = f(x) \text{ for some } x \in C\}$. We call $f(C)$ the *image* of $C$ under $f$. $f(A)$ is also called the *range* of $f$. Obviously $f(A) \subset B$, but it is not always the case that $f(A) = B$.

Let $V \subset B$, then $f^{-1}(V)$ denotes the set of all elements in $A$ such that $f(A) \in V$, i.e. $f^{-1}(V) = \{x \in A : f(x) \in V\}$. We call $f^{-1}(V)$ the *preimage* of $V$ under $f$.

**Injective, Surjective, Bijective, Inverse** The function $f$ is called *injective* or *one-to-one* if $\forall$ $b \in f(A)$, $\exists$ only one $x \in A$ with $b = f(x)$. In other words, a function is one-to-one if every output of the function has at most one input. Or, equivalently, for any $x, y \in A$, if $f(x) = f(y)$, then $x = y$.

The function $f$ is called *surjective* or *onto* if for each element $b \in B$ $\exists$ $x \in A$ such that $b = f(x)$. In other words, every element in $B$ is assigned to some element in $A$. This is identical to saying the range of $f$ is equal to the whole target space, $f(A) = B$.

A function $f$ from $A$ to $B$ is called *bijective* (or a bijection) if it is both injective and surjective. In this case, we can define the *inverse function* $f^{-1}$ from $B$ to $A$ such that $f^{-1}(f(x)) = x$ for all $x \in A$.

Notice that $C$ is not necessarily equal to $f^{-1}(f(C))$. If $f$ is onto, it is the case that $f(f^{-1}(V)) = V$. For example, let $f$ be a function from $\mathbb{R}$ to $\{1\}$ given by $f(x) = 1$, and let $C = \{2\}$. Then $f(C) = \{1\}$, $f^{-1}(\{1\}) = \mathbb{R}$, and $f(\mathbb{R}) = \{1\}$. So, $f(f^{-1}(\{1\}) = \{1\}$, and $f^{-1}(f(\{2\})) \neq \{2\}$.

**Exercises**

1. Show that the inverse function is not well defined if $f$ is not injective or not surjective.

2. Consider $f(x) = e^x$ mapping from $\mathbb{R}$ to $\mathbb{R}$. What is the range of $f$? What is the image of $[0, 1]$ under $f$? What is the preimage of $(0, 1]$? Is $f$ injective, surjective, or bijective? If it is bijective, what is the inverse function? If it is not bijective, can you restrict the domain to make it bijective and find the inverse function?

# 3 Functions in $\mathbb{R}$

Now we will consider functions that map from a subset of the real numbers into the set of real numbers. Here we can use the fact that in $\mathbb{R}$ we can measure the distance between points, i.e. the fact that $\mathbb{R}$ is a metric space. We will consider general metric spaces in the section on real analysis. In the following, let $X$ and $Y$ be two subsets of $\mathbb{R}$ and $f$ a function from $X$ to $Y$.

**Monotonicity** Let $x_1$ and $x_2$ be any two numbers in $X$ such that $x_1 < x_2$. $f$ is monotonically increasing if $f(x_1) \leq f(x_2)$ and monotonically decreasing if $f(x_1) \geq f(x_2)$. $f$ is said to be strictly monotonically increasing (decreasing) if $f(x_1) < f(x_2)$ $(f(x_1) > f(x_2))$.

**Boundedness** The function $f$ is bounded if $\exists$ $K \in \mathbb{R}$ such that $|f(x)| < K$ $\forall$ $x \in X$.

**Limit** We write $f(x) \to q$ as $x \to p$ or

$$\lim_{x \to p} f(x) = q$$

if for every $\epsilon > 0$ we can find $\delta > 0$ such that $|f(x) - q| < \epsilon$ for all $x$ for which $|x - p| < \delta$. $q$ is called the limit of $f$ at the point $p$.

**Continuity** The function $f$ is called continuous at the point $p \in X$ if

$$\lim_{x \to p} f(x) = f(p).$$

This is equivalent to saying that for every $\epsilon > 0$ we can find $\delta > 0$ such that $|f(x) - f(p)| < \epsilon$ for all $x$ for which $|x - p| < \delta$. $f$ is called continuous if it is continuous at every point of $X$.

**Concavity & Convexity** The function $f$ is *concave* if $\forall\ \alpha \in [0,1]$ and $x_1, x_2 \in \mathbb{R},\ f(\alpha x_1 + (1-\alpha)x_2) \geq \alpha f(x_1) + (1-\alpha)f(x_2)$. Graphically, this is a function where the chord drawn between any two points on its graph lies completely below or on the graph.

The function $f$ is *convex* if $\forall\ \alpha \in [0,1]$ and $x_1, x_2 \in \mathbb{R},\ f(\alpha x_1 + (1-\alpha)x_2) \leq \alpha f(x_1) + (1-\alpha)f(x_2)$. Graphically, this is a function where the chord drawn between any two points on its graph lies completely above or on the graph. This is not to be confused with a convex set.

**Exercise**

1. Which of the following functions is (1) monotonically increasing or decreasing, (2) bounded, (3) continuous, (4) concave or convex?
   (a) $f : \mathbb{R} \to \mathbb{R}, f(x) = x^2$
   (b) $f : \mathbb{R} \to \mathbb{R}, f(x) = x^3$
   (c) $f : \mathbb{R} \to \mathbb{R}, f(x) = \begin{cases} x^{-1} \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$
   (d) $f : \mathbb{R} \to \mathbb{R}, f(x) = e^x$
   (e) $f : \mathbb{R}_{++} \to \mathbb{R}, f(x) = \ln x$
2. Prove that $\lim_{x\to 0} x^2 = 0$. (In Proof section)
3. Show that $f(x) = \begin{cases} 1 \text{ if } x \geq 0 \\ -1 \text{ if } x < 0 \end{cases}$ is not continuous. (In Proof section)

# 4 Differentiation

## 4.1 Definition of the Derivative

**Definition** We say that a function $f$ is *differentiable at $x$* if

$$\frac{d}{dx}f(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

exists. If $f$ is differentiable for all $x$ in its domain, then we say $f$ is a *differentiable function.* $\frac{d}{dx}f(x)$ is called the *derivative* of $f$ at the point $x$. It is frequently also denoted as $f'(x)$.

**Interpretation** $\frac{d}{dx}f(x)$ is the change in the value of the function if $x$ changes infinitesimally. It is the slope of the line that is tangent to the graph of $f$ at point $x$.

**Example** Find the derivative of $f(x) = x^2$.

$$\frac{d}{dx}f(x) = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} f'(x) = \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h\to 0} \frac{2xh}{h} = 2x$$

## 4.2 Derivative Rules

Assume $k$ is an arbitrary constant, and that the two functions $u(x)$ and $v(x)$ are differentiable. Then the following rules hold:

1. Addition Rule
$$\frac{d}{dx}(u(x)+v(x)) = \frac{du}{dx} + \frac{dv}{dx}$$

2. Multiplicative Constant Rule
$$\frac{d}{dx}(k \cdot u(x)) = k \cdot \frac{du}{dx}$$

3. Product Rule
$$\frac{d}{dx}(u(x)\cdot v(x)) = \frac{du}{dx}\cdot v(x) + \frac{dv}{dx}\cdot u(x)$$

4. Quotient Rule
$$\frac{d}{dx}\left(\frac{u(x)}{v(x)}\right) = \frac{\frac{du}{dx}\cdot v(x) - \frac{dv}{dx}\cdot u(x)}{v(x)^2}$$

5. Power Rule
$$\frac{d}{dx}\left(x^k\right) = k \cdot x^{k-1}$$

6. Chain Rule
$$\frac{d}{dx}(u(v(x))) = \frac{d}{dv}u(v(x)) * \frac{d}{dx}v(x) = u'(v(x)) * v'(x)$$

Here are some formulas for the differentiation of special functions:

- $\frac{d}{dx}e^x = e^x$
- $\frac{d}{dx}\ln x = \frac{1}{x}$
- $\frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\cos x = -\sin x$

**Exercises**

1. Find the derivative of $\frac{3x^2}{1+\frac{4}{x}}$ with respect to $x$.

2. Prove as many of the derivative rules above as you can. Use Simon&Blume if you get stuck.

## 4.3 Higher Order Derivatives

Note that $\frac{d}{dx}f(x)$ is itself a function of $x$ - to every value of $x$ it assigns the slope of $f$ at that point. If $\frac{d}{dx}f(x)$ is a continuous function, we call $f$ *continuously differentiable*. The class of all continuously differentiable functions is called $C^1$.

If $\frac{d}{dx}f(x)$ is not only continuous but even differentiable, we can take the derivative of $\frac{d}{dx}f(x)$ and call it $\frac{d^2}{dx^2}$. It is also denoted as $f''(x)$ and called the *second order derivative*.

If the second order derivative is continuous, $f$ is said to be twice continuously differentiable. The class of all twice continuously differentiable functions is called $C^2$.

We can continue taking derivatives as long as they exist. The kth order partial derivative is denoted $\frac{d^k}{dx^k}$ or $f^{(k)}$. The class of all k-times continuously differentiable functions is called $C^k$.

**Example** Find the second order derivative of $f(x) = x^2$.
We have already computed the first order derivative: $\frac{d}{dx}f(x) = 2x$. Take the derivative of $2x$ by using the power rule above: $\frac{d^2}{dx^2}f(x) = 2$.

## 4.4 Critical Points, Mean Value Theorem, Sketching the graph of a function

Derivatives are helpful in determining the shape of a function. We will return to this when we look at optimization, but we'll cover the basic results already here.

**Definition** A function $f$ has a *local maximum* at a point $p$ if there exists $\delta > 0$ such that $f(q) \leq f(p)$ for all $q$ such that $|p - q| < \delta$.
Similarly, a function has a *local minimum* if there exists $\delta > 0$ such that $f(q) \geq f(p)$ for all $q$ such that $|p - q| < \delta$.

**Theorem** Let $f$ be a function defined on the interval $[a, b]$. If $f$ has a local maximum at a point $x \in (a, b)$ and if $f'(x)$ exists, then $f'(x) = 0$. The same is true for local minima.

This theorem says that *if* a function has a local maximum at a point and *if* its derivative exists at that point *then* the derivative has to be equal to zero at that point. A point $x$ at which $f'(x) = 0$ is called a *critical value* of $f$. It is a candidate for a local minimum or maximum, but it does not have to be one.

**Mean Value Theorem** Let $f$ be a continuous function on the interval $[a, b]$ and let $f$ be differentiable on $(a, b)$. Then there is a point $x \in (a, b)$ at which

$$f(b) - f(a) = (b - a)f'(x).$$

**Theorem** Let $f$ be differentiable on $(a, b)$. Then the following holds:

1. If $f'(x) \geq 0$ for all $x \in (a, b)$, then $f$ is monotonically increasing.
2. If $f'(x) = 0$ for all $x \in (a, b)$, then $f$ is constant.
3. If $f'(x) \leq 0$ for all $x \in (a, b)$, then $f$ is monotonically decreasing.

So the derivative of a function tells us whether a function slopes upwards, downwards, or is constant. If we want to know what the graph of a differentiable function $f$ looks like, we can start by determining the sign of $f'$. It is also helpful to look at $\lim_{x\to\pm\infty} f(x)$ and to compute $f$ at certain points, e.g. at 0 and at points where $f'$ changes signs.

**Exercise**

1. Prove the first theorem above.
2. Sketch the graph of $f(x) = x^3 + 2x^2 + 3$ using information on the sign of $f'$, $\lim_{x\to\pm\infty} f(x)$, and values of $f$ at selected points.

### 4.5 Taylor Polynomials

In this section we will look further into how derivatives can be used to approximate a function. As discussed above, $f'(a)$ is the slope of the line that is tangent to the graph of $f$ at point $a$. Approximating the function with its tangent, we can write

$$f(a+h) \approx f(a) + f'(a)h$$

Let us define difference between the left hand side and the right hand side as the remainder term

$$R(h,a) = f(a+h) - f(a) - f'(a)h.$$

Then by the definition of the derivative,

$$\frac{R(h,a)}{h} \to 0 \text{ as } h \to 0.$$

So not only does the remainder tend to 0 as $h$ gets smaller, it also tends to 0 *faster* than $h$.

We can approximate a differentiable function with a polynomial if we use higher order derivatives (for details why this works see SB Chapter 30.2). Define the *kth order Taylor polynomial* of a $C^k$ function $f$ around the point $a$ as

$$P_k(a+h) = f(a) + f'(a)h + \frac{1}{2!}f''(a)h^2 + \ldots + \frac{1}{k!}f'(a)h^k.$$

Define the difference between the two sides as the kth order remainder term

$$R_k(h,a) = f(a+h) - f(a) - f'(a)h - \frac{1}{2!}f''(a)h^2 - \ldots - \frac{1}{k!}f'(a)h^k$$

Then

$$\frac{R_k(h,a)}{h^k} \to 0 \text{ as } h \to 0.$$

In other words,

$$f(a+h) \approx P_k(a+h)$$

and the difference between the two sides tends to zero faster than $h^k$. For example, if $h = 0.1$ and $k = 3$, then $h^k = 0.001$ and so $P_k(a+h)$ gives an approximation of $f(a+h)$ that is exact up to the third decimal after the point. Notice how cool this is: We can approximate a function that is infinitely differentiable arbitrarily well using polynomials!

## 5 Integration

Differentiation was concerned with the slope of the graph of a function. Integration is concerned with the area under the graph of the function. The Fundamental Theorem of Calculus tells us that integration and differentiation are closely related: They are basically inverse operations.

### 5.1 Definite Integrals

Consider a function $f(x)$. The area under the graph of the function between points $x = a$ and $x = b$ is denoted by $\int_a^b f(x)dx$, and is called the definite integral of $f(x)$ between $a$ and $b$. If $f(t)$ and $g(t)$ are integrable functions, then the following properties of the definite integral hold:

1. $\int_a^b [f(t) + g(t)]\, dt = \int_a^b f(t)dt + \int_a^b g(t)dt$

2. $\int_a^b \lambda f(t)dt = \lambda \int_a^b f(t)dt$
3. $\int_a^c f(t)dt = \int_a^b f(t)dt + \int_b^c f(t)dt$
4. $\int_a^b f(t)dt = -\int_b^a f(t)dt$
5. $\int_a^a f(t)dt = 0$

## 5.2 Indefinite Integrals

If $f(x)$ is given then any function $F(x)$ such that $F'(x) = f(x)$ is called an indefinite integral of $f(x)$, or the anti-derivative. Note that there are infinitely many anti-derivatives of a function $f(x)$ since they can differ by a constant. We denote the anti-derivative by $\int f(x)dx$. The following are some simple rules for finding anti-derivatives:

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + C$
2. $\int e^x dx = e^x + C$
3. $\int a^x \ln(a)dx = a^x + C$
4. $\int f'(x)e^{f(x)}dx = e^{f(x)} + C$
5. $\int \frac{f'(x)}{f(x)}dx = \ln[f(x)] + C$

## 5.3 The Fundamental Theorem of Calculus

If $f(x)$ is continuous on $[a, b]$, and $F(x)$ is the anti-derivative of $f(x)$, then $\int_a^b f(x)dx = F(b) - F(a)$. This says that the integral of $f$ from $a$ to $b$ is nothing else than the anti-derivative of $F$ computed at $b$ minus the anti-derivative computed of $F$ computed at $a$.

**Example 1** Find the area under the curve $f(x) = x^2$ in the region $[1, 2]$.
By rule one given above, we know that the anti-derivative $F(x) = \frac{1}{3}x^3$. Therefore, the area under the curve is $F(b) - F(a) = \frac{1}{3} \cdot 2^3 - \frac{1}{3} \cdot 1^3 = \frac{7}{3}$.

**Example 2** Find the derivative of $f(x) = \frac{x^2}{(x+3)(x+2)}$.
We could use the quotient rule as described in the section on differentiation, but we would have to substitute $u = x^2$ and $v = (x + 3)(x + 2)$. This is likely to give us a big mess. But we can use rule five to calculate it easier.
By rule five,

$$\int \frac{f'(x)}{f(x)}dx = \ln[f(x)].$$

By the definition of an anti-derivative, we know that

$$\frac{d}{dx}\int \frac{f'(x)}{f(x)}dx = \frac{d}{dx}\ln[f(x)] = \frac{f'(x)}{f(x)},$$

which implies

$$f'(x) = f(x) \cdot \frac{d}{dx}\ln[f(x)].$$

It is easier to derivate the log of the function and multiply it by $f(x)$ than it is to use the quotient rule in this case.

$$f'(x) = \frac{x^2}{(x+3)(x+2)} \cdot \frac{d}{dx}\left(2\ln(x) - \ln(x+3) - \ln(x+2)\right) = \frac{x^2}{(x+3)(x+2)} \cdot \left(\frac{2}{x} - \frac{1}{x+3} - \frac{1}{x+2}\right)$$

$$= \frac{2x}{(x+3)(x+2)} - \frac{x^2}{(x+3)^2(x+2)} - \frac{x^2}{(x+3)(x+2)^2}$$

## 5.4 Strategies for Computing Integrals

Computing integrals for complicated functions can be tricky. Here are some methods that can be used to find the integrals of functions that are more complex. However, note that sometimes it can even be impossible to find an explicit expression for the integral.

### 5.4.1 Integration by Substitution

To find an integral of the form $\int_a^b f(g(x))g'(x)dx$ one can apply the substitution rule:

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

where $u = g(x)$ and $du = g'(x)dx$. (The proof of this rule uses the chain rule for derivatives and the FTC.)

**Example 1** Find $\int (x+1)^{10}dx$.
In this example, $f(x) = x^{10}$, $g(x) = x+1$ and $g'(x) = 1$. Let $u = x+1$. This implies $du = dx$. Substituting into the above equation, we now have to solve the easy integral $\int u^{10}du = \frac{u^{11}}{11}$. Substituting back in for $u$ we find that $\int (x+1)^{10}dx = \frac{(x+1)^{11}}{11}$. This is much easier than expanding the function $(x+1)^{10}$ and finding its integral.

**Example 2** Find $\int (3x+1)^{10}dx$.
In this example, $f(x) = x^{10}$, $g(x) = 3x+1$ and $g'(x) = 3$. We can still apply substitution by rewriting $\int (3x+1)^{10}dx = \frac{1}{3}\int (3x+1)^{10}3dx$ (the $\frac{1}{3}$ in front of the integral and the 3 inside the integral cancel!). Let $u = 3x+1$. This implies $du = 3dx$. Substituting into the above equation, we now have to solve the integral $\frac{1}{3}\int u^{10}du = \frac{u^{11}}{33}$. Substituting back in for $u$ we find that $\int (3x+1)^{10}dx = \frac{(3x+1)^{11}}{33}$.

**Example 3** Find $\int_e^{e^2} \frac{1}{x}\left[\ln(x)\right]^{-3} dx$
In this example, $f(x) = x^{-3}$, $g(x) = \ln x$ and $g'(x) = \frac{1}{x}$. Let $u = \ln(x)$. Therefore $du = \frac{1}{x}dx$, the upper limit of integration is $\ln(e^2) = 2$, and the lower limit of integration is $\ln(e) = 1$. The new integral reads

$$\int_1^2 u^{-3}du = -\frac{1}{2}u^{-2}|_1^2 = -\frac{1}{2}[2^{-2} - 1^{-2}] = \frac{3}{8}.$$

**Exercises**

1. Find $\int \frac{20}{(4-5x)^3}dx$.
2. Find $\int_1^3 \frac{e^{\sqrt{x}}}{\sqrt{x}}dx$.
3. Find $\int_1^3 4xe^{x^2}dx$

### 5.4.2 Integration by Parts

To find an integral of the form $\int_a^b f(x) \cdot g'(x)dx$ one can apply the "integration by parts" rule:

$$\int f(x) \cdot g'(x)dx = f(x) \cdot g(x) - \int g(x) \cdot f'(x)dx$$

**Example 1** Find $\int \ln(x)dx$.
Let $f(x) = \ln(x)$ and $g'(x) = 1$. Then $f'(x) = \frac{1}{x}dx$ and $g(x) = x$. Then

$$\int f(x) \cdot g'(x)dx = f(x) \cdot g(x) - \int g(x) \cdot f'(x)dx = x \cdot \ln(x) - \int x \cdot \frac{1}{x}dx = x[\ln(x) - 1].$$

**Example 2** Find $\int xe^{-x}dx$.
Let $f(x) = x \Rightarrow f'(x) = 1$. Let $g'(x) = e^{-x}dx \Rightarrow g(x) = -e^{-x}$. Then

$$\int f(x) \cdot g'(x)dx = f(x) \cdot g(x) - \int g(x) \cdot f'(x)dx = -xe^{-x} - \int -e^{-x}dx = -xe^{-x} - e^{-x} = -e^{-x}(1+x).$$

**Exercises**

1. Find $\int x^2 \ln(x)dx$
2. Find $\int_0^1 x^3 e^{4x}dx$

## 6 Homework

Differentiate the following:

1. $f(x) = x^2 + 3x - 4$
2. $y = x^{-\frac{2}{3}}$
3. $g(x) = x^2 + x^{-3}$
4. $y = \frac{x^2+4x+3}{\sqrt{x}}$
5. $f(x) = x^2 e^x$
6. $V(x) = (2x+3)\left(x^4 - 2x\right)$
7. $f(x) = \frac{x}{x+\frac{c}{x}}$
8. $F(y) = \left(\frac{1}{y^2} - \frac{3}{y^4}\right)\left(y + 5y^3\right)$
9. $y = e^{\sqrt{x}}$
10. $y = e^{ax^2+bx+c}$
11. $y = \ln(t+9)$
12. $y = \ln(x) - \ln(1+x)$

Find the following:

1. $\int 8x^{-5} dx,\ x \neq 0.$
2. $\int (7e^x + 3) dx$
3. $\int \frac{6x}{x^2+13} dx$
4. $\int (x+3)(x+1)^{\frac{1}{2}}$
5. $\int xe^x dx$
6. $\int x^3\sqrt{1+x^2} dx$

Evaluate the following:

1. $\int_0^1 x(x^2+6) dx$
2. $\int_{-1}^1 (ax^2 + bx + c) dx$
3. $\int_1^2 e^{-2x} dx$