AP6101
9/22/05

SOLUTIONS #2

QUESTION #1 PROBLEM 3.1

$\bar{B} = B_0 \hat{z}$, $\bar{u} = V_E \hat{x}$, POSITIVELY CHARGED PARTICLE

a) $\bar{u} = \bar{E} \times \bar{B} / B^2$, SO $\bar{B} \times \bar{u} = \bar{E}$, AND $\bar{E} = (\hat{z} \times \hat{x}) B_0 V_E = \hat{y} B_0 V_E$

b) AT $t = 0$, $(x, y, z) = (0, 0, 0)$ AND $(U_x, U_y, U_z) = (0, 0, 0)$

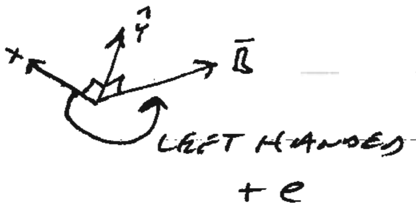
THEN, $V_z(t) = 0 \quad z(t) = 0$

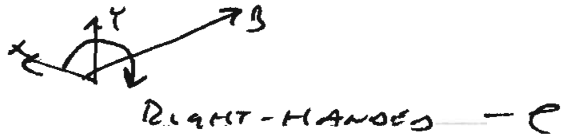
$V_y(t) = -\rho \omega_c \cos(\omega_c t + \varphi) \quad y(t) = Y_0 - \rho \sin(\omega_c t + \varphi)$

$V_x(t) = V_E - \rho \omega_c \sin(\omega_c t + \varphi) \quad x(t) = V_E t + \rho \cos(\omega_c t + \varphi)$

WHERE $\omega_c = qB/m$

NOTE: FOR A POSITIVE CHARGE, CYCLOTRON MOTION IS "LEFT-HANDED"!!





QUESTION: WHAT ARE Y_0, ρ, AND φ?

ANSWER: MATCH BOUNDARY/INITIAL VALUES...

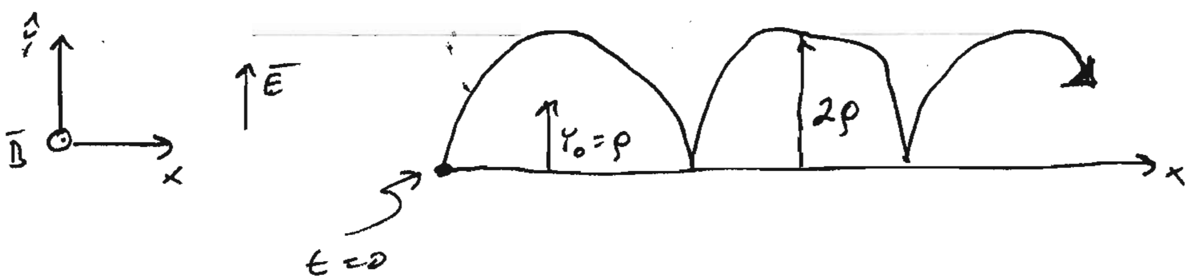
AT $t = 0$, $x = 0 = \cos(\varphi)$

$y = 0 = Y_0 - \rho \sin(\varphi)$

$V_x = 0 = V_E - \rho \omega_c \sin(\varphi)$

$V_y = 0 = \cos(\varphi)$

THUS, $\varphi = \frac{\pi}{2}$, $Y_0 = \rho$, AND $\rho = V_E / \omega_c$



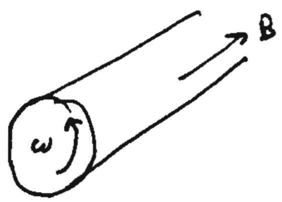
QUESTION #1 (CONTINUED)

c) ENERGY = K.E. + P.E.

$$K.E. = \frac{1}{2} m \left(v_x^2 + v_y^2\right) = \frac{1}{2} m \left(2V_E^2 - 2V_E^2 \sin\left(\omega_c t + \frac{\pi}{2}\right)\right)$$
$$= m V_E^2 \left(1 - \sin\left(\omega_c t + \frac{\pi}{2}\right)\right)$$
$$P.E. = -eV_E B_0 y = -eV_E B_0 \left(y_0 - \rho \sin\left(\omega_c t + \frac{\pi}{2}\right)\right)$$
$$= -\frac{e V_E^2 B_0}{\omega_c} \left(1 - \sin\left(\omega_c t + \frac{\pi}{2}\right)\right)$$
$$= -m V_E^2 \left(1 - \sin\left(\omega_c t + \frac{\pi}{2}\right)\right)$$

THUS, TOTAL ENERGY = 0

QUESTION #2 PROBLEM 3.2



a)
$$\bar{V}_E = \omega_0 r \left(\hat{z} \times \hat{r}\right) = \frac{\bar{E} \times \bar{B}}{B^2} = \frac{E_r}{B}\left(\hat{r} \times \hat{z}\right)$$
$$\therefore \quad E_R = -\omega_0 r B_0$$

b) $$\rho = \epsilon_0 \nabla \cdot \bar{E} = -\epsilon_0 \frac{\omega_0 B_0}{r} \frac{\partial}{\partial r}\left(r^2\right) = -2\epsilon_0 \omega_0 B_0$$

c) $$\bar{E} = -\nabla \Phi \quad \therefore \quad \frac{\partial \Phi}{\partial r} = \omega_0 B_0 r \Rightarrow \Phi(r) = \omega_0 B_0 \frac{r^2}{2}$$

d) IF THERE WERE A UNIFORM CHARGE DENSITY, THEN THE COLUMN ROTATES AS A RIGID ROTOR. IMPOSING ROTATION, IN GENERAL, INVOLVES PLASMA TRANSPORT PROPERTIES AND BOUNDARY/EDGE EFFECTS.

QUESTION #3 PROBLEM 3.4

a) TWO METHODS TO DETERMINE A FIELD-LINE

METHOD #1

$$\frac{dr}{B_r} = \frac{r\,d\theta}{B_\theta} = \frac{ds}{|B|}$$

THUS

$$\int \frac{dr}{r} = 2\int \frac{\cos\theta\, d\theta}{\sin\theta} = 2\int \frac{d(\sin\theta)}{\sin\theta}$$

$$\ln r = 2\ln(\sin\theta)$$

OR $$r = R\sin^2\theta$$

METHOD #2

$$\bar{B} = \nabla\varphi \times \nabla\psi$$ SO (ψ, φ) LABEL A FIELD-LINE

SINCE $\nabla\varphi = \frac{\hat{\varphi}}{r\sin\theta}$, WE WRITE

$$\hat{\theta}\cdot\bar{B} = \hat{\theta}\cdot(\nabla\varphi\times\nabla\psi) = (\hat{\theta}\times\hat{\varphi})\cdot\frac{1}{r\sin\theta}\left(\frac{\partial\psi}{\partial r}\hat{r} + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\theta}\hat{\theta}\right)$$

$$= \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}$$

$$= \frac{\mu_0 m}{4\pi r^3}\sin\theta$$

SOLVING FOR $\psi(r)$ ... $\psi(r,\theta) = -\frac{\mu_0 m}{4\pi r}\sin^2\theta$

SO $\frac{\sin^2\theta}{r}$ = CONSTANT ALONG A FIELD LINE

b) $\bar{K} = (\hat{b}\cdot\bar{\nabla})\hat{b} \propto \frac{1}{R_c}$ $\hat{b} = \frac{2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}}{3\cos^2\theta + 1}$

IN SPHERICAL COORDINATES

$$\hat{r}\cdot\bar{K}\Big|_{\theta\to\frac{\pi}{2}} = \frac{\hat{b}_\theta}{r}\frac{\partial b_r}{\partial\theta} - \frac{\hat{b}_\theta^2}{r} = \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{2\cos\theta}{3\cos^2\theta+1}\right)\Bigg|_{\theta=\frac{\pi}{2}} - \frac{1}{r}$$

$$= \frac{1}{r}\left[-2-1\right] = -\frac{3}{r}$$ QED

PROBLEM #3.4 (CONTINUED)

NOTE ALSO $\quad \nabla B = \hat{r}\dfrac{\partial B}{\partial r} + \dfrac{\hat{\theta}}{r}\dfrac{\partial B}{\partial \theta}$

$\dfrac{\partial B}{\partial r} = -\dfrac{3B}{r}$. THEREFORE $\quad \hat{r}\cdot\bar{\kappa} = \hat{r}\cdot\nabla B/B$

c) $\bar{V}_x = \dfrac{2W_\parallel}{qB}\left[\dfrac{\hat{b}\times(-\hat{r}\,R_c)}{|R_c|^2}\right] = \dfrac{2W_\parallel}{qB}\left[\hat{b}\times\bar{\kappa}\right]$

$= \dfrac{2W_\parallel}{qB}\left(\hat{\theta}\times\left(-\hat{r}\dfrac{3}{r}\right)\right)$ At $\theta = \frac{\pi}{2}$

$= \hat{\varphi}\,\dfrac{6W_\parallel}{qBr}$

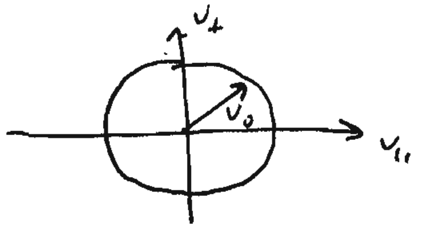
d) $\bar{V}_{\nabla B} = \dfrac{W_\perp}{qB}\left[\dfrac{\hat{b}\times\nabla B}{B}\right] = \hat{\varphi}\,\dfrac{3W_\perp}{qBr}$ @ $\theta = \frac{\pi}{2}$

e) $\dfrac{V_x}{V_{\nabla B}} = \dfrac{2W_\parallel}{W_\perp}$

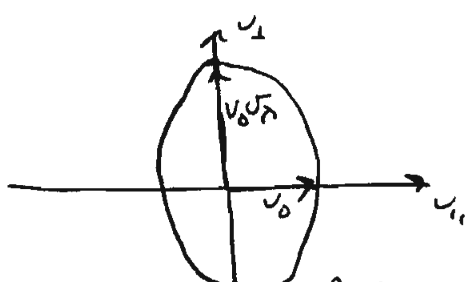
QUESTION #4

THIS IS A VELOCITY-SPACE PROBLEM RECOGNIZING THAT $\mu = \frac{1}{2}v_\perp^2/B$ = CONSTANT DURING ADIABATIC CHANGES IN THE MAGNETIC FIELD.

FOR AN ADIABATIC INCREASE IN B, THEN $v_\perp' \to \sqrt{\lambda}\, v_\perp$.



BEFORE



AFTER

QUESTION #4 (CONT.)

IN ORDER TO SOLVE THIS PROBLEM, IT IS USEFUL TO PRACTICE INTEGRATING VELOCITY-SPACE DISTRIBUTION FUNCTIONS. FOR SIMPLICITY, WE'LL NORMALIZE THE DENSITY AND PARTICLE MASS TO 1: $n = m = 1$. THEN, THE DISTRIBUTION FUNCTION HAS THE FOLLOWING PROPERTIES

$$\iiint d^3v \, f = 1$$
$$\iiint d^3v \, \tfrac{1}{2} v^2 f = \langle \tfrac{1}{2} v^2 \rangle = \text{K.E.}$$
$$\iiint d^3v \, \tfrac{1}{2} v_\perp^2 f = \langle \tfrac{1}{2} v_\perp^2 \rangle = \text{PERPENDICULAR ENERGY}$$
$$\iiint d^3v \, \tfrac{1}{2} v_\parallel^2 f = \langle \tfrac{1}{2} v_\parallel^2 \rangle = \text{PARALLEL ENERGY.}$$

LET'S LOOK AT THE INITIAL DISTRIBUTION FUNCTION

$$f(v) = N \, \delta(v - v_0)$$ WHERE N = NORMALIZATION AND $\delta(\cdots)$ IS THE FAMOUS "DIRAC" DELTA FUNCTION

THIS DISTRIBUTION IS A SPHERE OF RADIUS v_0 IN VELOCITY SPACE. THE NORMALIZATION IS

$$\iiint d^3v \, f = 1 = \int_0^\infty 4\pi v^2 dv \, N \, \delta(v - v_0)$$
$$= 4\pi v_0^2 N \quad \therefore \quad N = \frac{1}{4\pi v_0^2}$$

WE ALSO HAVE $\langle \tfrac{1}{2} v^2 \rangle = \tfrac{1}{2} v_0^2$ AS EXPECTED.

TO FIND THE PERPENDICULAR KINETIC ENERGY, WE WRITE IN CYLINDRICAL VELOCITY-SPACE

$$\iiint d^3v \, \tfrac{1}{2} v_\perp^2 f = \langle \tfrac{1}{2} v_\perp^2 \rangle = \int_0^\infty 2\pi v_\perp dv_\perp \int_{-\infty}^{\infty} dv_\parallel \, \tfrac{1}{2} v_\perp^2 \, \frac{\delta(\sqrt{v_\perp^2 + v_\parallel^2} - v_0)}{4\pi v_0^2}$$

QUESTION #4 (CONT.)

REMEMBER THE PROPERTY OF DIRAC DELTA FUNCTIONS:

$$\int_{-\infty}^{\infty} dx\, h(x)\, \delta(g(x)) = \int_{-\infty}^{\infty} dx\, h(x)\, \delta(g'(x-x_0)) \quad \text{WHERE } g(x_0)=0$$

$$= \frac{h(x_0)}{|g'|} \qquad g' = \left.\frac{dg}{dx}\right|_{x=x_0}$$

THEREFORE,

$$\langle \tfrac{1}{2} v_\perp^2 \rangle = \int_{-v_0}^{+v_0} dv_\parallel \frac{1}{2v_0} \frac{1}{2}(v_0^2 - v_\parallel^2) = \frac{2}{3}\left(\frac{1}{2} v_0^2\right)$$

WE ALSO HAVE $\langle \tfrac{1}{2} v_\parallel^2 \rangle = \int_{-v_0}^{v_0} dv_\parallel \frac{1}{2v_0} \frac{1}{2} v_\parallel^2 = \frac{1}{3}\left(\frac{1}{2} v_0^2\right)$

NOTICE $\langle \tfrac{1}{2} v^2 \rangle = \langle \tfrac{1}{2} v_\perp^2 \rangle + \langle \tfrac{1}{2} v_\parallel^2 \rangle$ AS EXPECTED.

NOW, WE CAN COMPUTE WHAT HAPPENS AFTER AN ADIABATIC COMPRESSION WHERE $v_\perp \longrightarrow \sqrt{\lambda}\, v_\perp$.
THE DISTRIBUTION HAS THE FORM

$$f(v) = N\, \delta\left(\sqrt{\frac{v_\perp^2}{\lambda} + v_\parallel^2} - v_0\right)$$

FIRST, WE NORMALIZE:

$$1 = \int_0^{\infty} 2\pi v_\perp dv_\perp \int_{-\infty}^{\infty} dv_\parallel\, N\, \delta\left(\sqrt{\frac{v_\perp^2}{\lambda} + v_\parallel^2} - v_0\right)$$

$$= 4\pi v_0^2 \lambda N \quad \therefore \quad N = \frac{1}{4\pi v_0^2 \lambda}$$

ON THE NEXT PAGE, WE COMPUTE $\langle \tfrac{1}{2} v_\perp^2 \rangle$

QUESTION #4 (CONT.)

$$K.E.\ (\text{AFTER COMPRESSION}) = \int_0^\infty 2\pi v_\perp\, dv_\perp \int_{-\infty}^{\infty} dv_\parallel\ \tfrac{1}{2}\left(v_\perp^2 + v_\parallel^2\right) f$$

$$= \frac{1}{2v_0^2\lambda}\int_0^\infty v_\perp\, dv_\perp \int_{-\infty}^{\infty} dv_\parallel\ \tfrac{1}{2}\left(v_\perp^2+v_\parallel^2\right)\delta\left(\sqrt{\tfrac{v_\perp^2}{\lambda}+v_\parallel^2}-v_0\right)$$

LET'S CHANGE VARIABLES: $\tilde{v}_\perp^2 \equiv \dfrac{v_\perp^2}{\lambda}$ so $v_\perp dv_\perp = \lambda\, d\tilde{v}_\perp \tilde{v}_\perp$

$$K.E. = \frac{1}{2v_0^2}\int_0^\infty \tilde{v}_\perp\, d\tilde{v}_\perp \int_{-\infty}^{\infty} dv_\parallel\ \tfrac{1}{2}\left(\lambda\tilde{v}_\perp^2 + v_\parallel^2\right)\delta\left(\sqrt{\tilde{v}_\perp^2+v_\parallel^2}-v_0\right)$$

$$= \frac{1}{2v_0^2}\int_0^\infty \tilde{v}_\perp\, d\tilde{v}_\perp \int_{-\infty}^{\infty} dv_\parallel\left(\tfrac{1}{2}v^2 + (\lambda-1)\tfrac{1}{2}\tilde{v}_\perp^2\right)\delta\left(\sqrt{\tilde{v}_\perp^2+v_\parallel^2}-v_0\right)$$

$$= \langle \tfrac{1}{2}v^2\rangle + (\lambda-1)\langle\tfrac{1}{2}\tilde{v}_\perp^2\rangle = \tfrac{1}{2}v_0^2 + (\lambda-1)\tfrac{2}{3}\left(\tfrac{1}{2}v_0^2\right)$$

THE REST IS EASY! !!!

0. INITIAL K.E. $= \frac{1}{2}v_0^2$
1. AFTER ADIABATIC COMPRESSION, NEW K.E. IS
$$K.E. \to \tfrac{1}{2}v_0^2\left[1 + \tfrac{2}{3}(\lambda-1)\right]$$
2. AFTER VELOCITY DISTRIBUTION BECOMES ISOTROPIC, THE K.E. <u>DOES</u> NOT CHANGE
3. AFTER ADIABATIC DE-COMPRESSION NEW K.E. IS
$$K.E. \to \tfrac{1}{2}v_0^2\left[1+\tfrac{2}{3}(\lambda-1)\right]\left[1+\tfrac{2}{3}\left(\tfrac{1}{\lambda}-1\right)\right]$$
4. AFTER ISOTROPIC, K.E. REMAINS UNCHANGED

THE FINAL K.E. IS
$$K.E.\ (\text{FINAL}) = \tfrac{1}{2}v_0^2\left[1+\tfrac{2}{3}(\lambda-1)\right]\left[1-\tfrac{2}{3}\frac{(\lambda-1)}{\lambda}\right]$$
$$= \tfrac{1}{2}v_0^2\left[1+\tfrac{2}{9}\frac{(\lambda-1)^2}{\lambda}\right]$$

FINALLY, THE MAXIMUM ENERGY IS $E_{max} = \frac{1}{2}v_0^2\lambda$ THE MINIMUM ENERGY IS $\frac{1}{2}v_0^2\frac{1}{\lambda}$.
NOTICE THAT K.E. (FINAL) $\propto \lambda$ AS $\lambda \to \infty$.