9/28/05

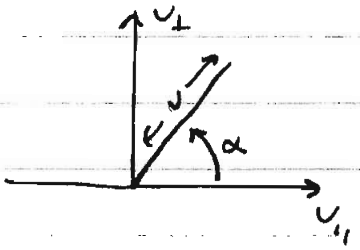
PLASMA I HW SOLUTIONS #3

Q1) P. 3.7 $B = B_z = B_0\left(1 + \frac{z^2}{L^2}\right)$

$$\frac{dv_\parallel}{dt} = \frac{d^2 z}{dt^2} = -\frac{\mu}{m}\frac{\partial B}{\partial z} = -\frac{2\mu B_0}{mL^2} z$$

THUS, $\omega_B^2 = \frac{2\mu B_0}{mL^2}$ AND $z(t) = z_m \cos(\omega_B t + \varphi)$

DEFINE α:



$\sin\alpha = \frac{v_\perp}{v}$ THUS $\mu B_0 = W \sin^2\alpha$

AND

$$\omega_B^2 = \frac{2W}{mL^2}\sin^2\alpha$$

Q2) P. 3.9 $(\mu, J) = \text{CONSTANT}$

$\mu = \frac{1}{2} m v_\perp^2 / B_0$

$J = 2 v_\parallel L$ INITIALLY $\frac{v_\perp}{v_\parallel} = 1 = \frac{\sqrt{2\mu B_0/m}}{(J/2L)}$

$$\text{ENERGY} = \tfrac{1}{2} m v_\perp^2 + \tfrac{1}{2} m v_\parallel^2 = \mu B_0 + J^2 m / 8L^2$$

$$\frac{W_f}{W_i} = \frac{\mu B_0 + mJ^2/8(L/2)^2}{\mu B_0 + mJ^2/8L^2}$$ $\mu B_0 = mJ^2/8L$

$$= \frac{1 + 4}{2} = \frac{5}{2}$$

AS $L \to$ GET'S SMALLER, THEN $v_\parallel$ GETS LARGER

Q3) P. 3.10 $(\mu, J) = \text{CONSTANT}$

$\mu = \frac{1}{2} m v_\perp^2(0) / B_0$ $J = \oint ds\, v_\parallel(s) = \int dt\, v_\parallel^2(t)$

$v_\parallel^2(t) = z_m^2 \omega_B^2 \sin^2(\omega_B t)$; $\omega_B^2 = \frac{2\mu B_0}{mL^2}$

PROBLEM 3.10 (CONT.)

THUS, $J = \oint dt\, v_\parallel^2(t) = \pi z_m^2 \omega_B = \frac{2\pi}{m}\left(\frac{1}{2} v_\parallel^2(0)\right)/\omega_B$

TO DETERMINE THE MAXIMUM B...

$$\mu B_m = \mu B_0 + \frac{1}{2} m v_\parallel^2 = \mu B_0 + \frac{m}{2\pi} J \omega_B$$

OR

$$\frac{B_m}{B_0} = 1 + \frac{J}{2\pi L}\sqrt{\frac{m}{\mu B_0}}$$

INITIALLY, $\frac{B_m}{B_0} = 2 \Rightarrow \frac{J}{\sqrt{\mu}} = 2\pi L \sqrt{B_0/m}$

ALSO, $\left(\frac{z_m}{L}\right)^2 = \frac{B_m}{B_0} - 1 = \frac{J}{2\pi L}\sqrt{\frac{m}{\mu B_0}}$

PART A) IF B_0 INCREASES BY 2X, THEN

$$\frac{B_m}{B_0} = 1 + \sqrt{\frac{B_0}{B_{NEW}}} = 1 + \frac{\sqrt{2}}{2} = 1.7$$

$$\left(\frac{z_m}{L}\right)^2 = \frac{\sqrt{2}}{2} \Rightarrow z_m/L = 0.84$$

PART B) IF L IS NOW DECREASED BY A FACTOR OF 2

$$\frac{B_m}{B_0} = 1 + \frac{L}{L_{NEW}}\sqrt{\frac{B_0}{B_{NEW}}} = 2.41$$

$$\left(\frac{z_m}{L}\right)^2 = \frac{L}{L_{NEW}}\sqrt{\frac{B_0}{B_{NEW}}} \Rightarrow \frac{z_m}{L} = 1.19$$

Q4) P. 4.1

$$f(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, e^{-i(k-k_0)x}\, e^{-a^2x^2/2}$$

LET $\tilde{k} = k - k_0$ $\quad f(\tilde{k}) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, e^{-i\tilde{k}x}\, e^{-a^2x^2/2}$

[illegible] ... THE SQUARE ...

Q4 (CONT.)

$$\text{SO}\quad f(\tilde{k}) = \frac{e^{-k^2/2a^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, e^{-\left(\frac{ax}{\sqrt{2}} + \frac{ik}{a\sqrt{2}}\right)^2}$$

$$= \frac{1}{a} e^{-k^2/2a^2}$$

Q5 | P. 4.2 DEFINE:

$$\langle \Delta x^2 \rangle \equiv \frac{\int_{-\infty}^{\infty} dx\, \frac{1}{2} x^2 |f| dx}{\int_{-\infty}^{\infty} dx\, |f| dx} \quad \text{WHERE } |f(x)| = e^{-a^2x^2/2}$$

$$\langle \Delta k^2 \rangle \equiv \frac{\int_{-\infty}^{\infty} dk\, \frac{1}{2}(k-k_0)^2 |f|}{\int_{-\infty}^{\infty} dk\, |f|} \quad |f(k)| = \frac{1}{a} e^{-(k-k_0)^2/2a^2}$$

THEN, WITH $\int_{-\infty}^{\infty} dx\, |f| = \frac{\sqrt{2\pi}}{a}$; $\int_{-\infty}^{\infty} dk\, |f| = \sqrt{2\pi}$,

WE GET

$$\langle \Delta x^2 \rangle \sim \frac{1}{2a^2} \quad \langle \Delta k^2 \rangle \sim \frac{a^2}{2} \quad \text{AND } \langle \Delta x^2 \rangle \langle \Delta k^2 \rangle = \frac{1}{4}$$

QUESTION 6 PART a]

$$f_{po} = \omega_{po}/2\pi = 8.98 \sqrt{n}\ \text{kHz}$$
$$= 2.8\ \text{MHz FOR } n \sim 10^{10}\ \text{cm}^{-3}$$

$f < f_{po}$ ARE REFLECTED.

PART b] ASSUME AN ISOTROPIC TRANSMITTING ANTENNA. THEN THE RECIEVED POWER IS THE SQUARE OF THE SUM OF THE FIELDS FROM DIRECT AND INDIRECT PATHS.

LET'S DO A SIMPLE LIMIT, $L < 2h$ ($h = 100\text{km}$)

Q6 (CONT.)

IN THIS LIMIT

$$\text{POWER RECEIVED} \simeq \left| \frac{1}{\sqrt{4\pi L^2}} e^{jkL} + \frac{1}{\sqrt{4\pi(L^2+4h^2)}} e^{jk\sqrt{L^2+4h^2}} \right|^2$$

$$= \frac{1}{4\pi L^2} \left| 1 + \sqrt{\frac{L^2}{L^2+4h^2}}\, e^{jk\left(\sqrt{L^2+4h^2} - L\right)} \right|^2$$

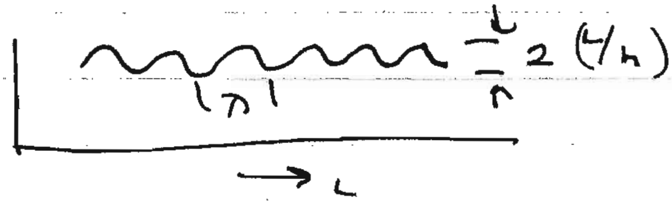
IF $2h >> L$, THEN

$$P_{REC} = \frac{1}{4\pi^2 L^2} \left| 1 + \frac{L}{2h} e^{jk\left(2h\left(1+\frac{L^2}{4h^2}\right) - L\right)} \right|^2$$

$$\simeq \frac{1}{4\pi L^2} \left| 1 + \frac{L}{2h} e^{jk(2h-L)} \right|^2$$

$$\simeq \frac{1}{4\pi L^2} \left(1 + \left(\frac{L}{2h}\right)^2 + \frac{L}{h}\cos\left(k(2h-L)\right) \right)$$

P_{REC}

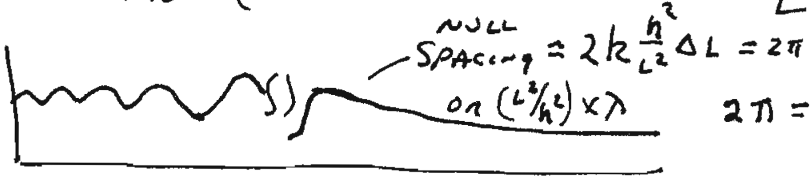


THE POWER MODULATES LIKE THE WAVELENGTH.

IF $2h << L$, THEN

$$P_{REC} \simeq \frac{1}{4\pi L^2} \left| 1 + e^{jk\left(L\left(1+2h^2/L^2\right) - L\right)} \right|^2$$

$$\simeq \frac{1}{4\pi L^2} \left(1 + 1 + 2\cos\left(\frac{2kh^2}{L}\right) \right)$$



$$2\pi = \frac{2kh^2}{L_1} - \frac{2kh^2}{L_2} = \frac{2kh^2}{L_1 L_2}\Delta L$$