10/13/05

APPH 6101 SOLUTIONS #5

4.14 THE GROUP VELOCITY IS $\frac{\partial\omega}{\partial\bar{k}} = \hat{k}\frac{\partial\omega}{\partial k} + \frac{\hat{\theta}}{k}\frac{\partial\omega}{\partial\theta}$

WHERE $\omega(k,\theta) = c^2 k^2 (\omega_c/\omega_p^2)\cos\theta$

$$\frac{\partial\omega}{\partial k} = \frac{2\omega}{k} \qquad \frac{\partial\omega}{\partial\theta} = -\frac{\sin\theta}{\cos\theta}\,\omega$$

SO $\bar{v}_g = \hat{k}\frac{2\omega}{k} - \hat{\theta}\frac{\omega}{k}\frac{\sin\theta}{\cos\theta} = \frac{\omega}{k}\left(2\hat{k} - \hat{\theta}\frac{\sin\theta}{\cos\theta}\right)$

TO FIND THE ANGLE, ψ, WITH RESPECT TO THE MAGNETIC FIELD, WE USE THE DEFINITIONS

$$\sin\psi = \frac{|\hat{b}\times\bar{v}_g|}{|v_g|} \qquad \cos\psi\ \frac{\hat{b}\cdot\bar{v}_g}{|v_g|}$$

WHERE $\hat{b}\times\hat{k} \propto \sin\theta \qquad \hat{b}\cdot\hat{k} \propto \cos\theta$

$\hat{b}\times\hat{\theta} \propto \cos\theta \qquad \hat{b}\cdot\hat{\theta} \propto -\sin\theta$

THEN $\tan\psi = \frac{\sin\psi}{\cos\psi} = \frac{|\hat{b}\times\bar{v}_g|}{\hat{b}\cdot\bar{v}_g} \sim \frac{2\sin\theta - \sin\theta}{2\cos\theta + \frac{\sin^2\theta}{\cos\theta}}$

$$= \frac{\sin\theta\cos\theta}{1+\cos^2\theta}$$

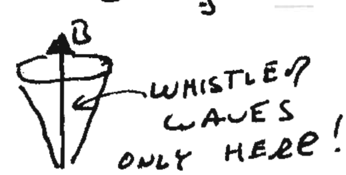
NOW, TAKING $\frac{d}{d\theta}\tan\psi = \frac{1}{\cos^2\psi}\frac{d\psi}{d\theta}$, WE FIND

$\frac{d\psi}{d\theta} = 2\cos^2\psi\,\frac{1+3\cos(2\theta)}{(3+\cos(2\theta))^2}$. THE <u>MAXIMUM</u> ANGLE

IS $\cos 2\theta = -\frac{1}{3}$ OR $\cos^2\theta - \sin^2\theta = -\frac{1}{3}$

OR $\sin^2\theta = \frac{1}{3}$ OR $\cos^2\theta = \frac{2}{3}$

ANS $\tan\psi = \frac{1}{2\sqrt{2}} \Rightarrow \psi_{MAX} = 19.5^\circ$



4.17 Proceeding as in class...

$$R = 1 - \frac{\omega_{pi}^2}{\omega(\omega+\omega_{ci})} - \frac{\omega_{pe}^2}{\omega(\omega-\omega_{ce})}$$

$$\approx 1 - \frac{\omega_{pi}^2}{\omega\,\omega_{ci}}\left(1 - \frac{\omega}{\omega_{ci}} + \cdots\right) + \frac{\omega_{pe}^2}{\omega\,\omega_{ce}}\left(1 - \frac{\omega}{\omega_{ce}} + \cdots\right.$$

$$\approx 1 + \frac{\omega_{pi}^2}{\omega_{ci}^2}$$

So $R = L = S$ and $D = 0$ $\quad P = 1 - \frac{\omega_{pe}^2}{\omega^2} \to -\infty$

$$\mathrm{DET}\begin{pmatrix} S - n^2\cos^2\theta & 0 & n^2\sin\theta\cos\theta \\ 0 & S - n^2 & 0 \\ n^2\sin\theta\cos\theta & 0 & P - n^2\sin^2\theta \end{pmatrix} \to 0$$

or

$$(S-n^2)\left(PS - n^2\left(S\sin^2\theta + P\cos^2\theta\right)\right) = 0$$

Two Alfven modes...

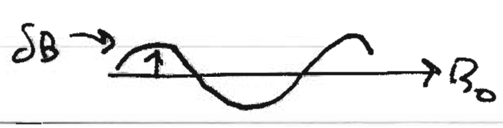
$$n^2 = S \qquad n^2 = \frac{PS}{S\sin^2\theta + P\cos^2\theta} \approx \frac{S}{\cos^2\theta}$$

But $S = 1 + \frac{\omega_{pi}^2}{\omega_{ci}^2} \approx \frac{e^2 n_0}{\epsilon_0 M_i}\frac{M_i^2}{e^2 B_0^2} = \frac{n_0 M_i}{\epsilon_0\mu_0\left(B_0^2/\mu_0\right)} = \frac{c^2}{V_A^2}$

4.18) $\lambda\frac{d^2y}{dt^2} = T\frac{d^2y}{dz^2} \Rightarrow y(z,t) \sim e^{-j(\omega t - kz)}$

where $\omega^2 = k^2(T/\lambda)$ or $v = \sqrt{T/\lambda}$

For shear Alfven waves, we have $\omega = k V_A$



where (plasma/field perturbation) propagates!

#4.18 CONT.) TO SEE THIS, WE TAKE

$$\bar{k} = \hat{z}\, k$$
$$\bar{E} = \hat{x}\, E_x$$
$$\bar{k} \times \bar{E} = \omega\, \delta \bar{B}$$

OR $\delta B_Y = \dfrac{k}{\omega} E_X$

MAXWELL'S EQUATIONS

$$\nabla \times (\nabla \times \delta \bar{B}) = -\frac{1}{c^2}\frac{\partial^2 \delta B}{\partial t^2} + \nabla \times \mu_0 \bar{J}$$

BUT $\mu_0 \bar{J} = \mu_0 \bar{\bar{\sigma}} \cdot \bar{E}$ WHERE

$$\mu_0 \bar{\bar{\sigma}} = \frac{1}{c^2}\begin{bmatrix} -i\omega \dfrac{\omega_{pi}^2}{\omega_{ci}^2} & 0 & 0 \\ 0 & -i\omega \dfrac{\omega_{pi}^2}{\omega_{ci}^2} & 0 \\ 0 & 0 & \dfrac{i}{\omega} \end{bmatrix} \quad \text{as } \omega \to 0$$

THEREFORE

$$\mu_0 \bar{\bar{\sigma}} \cdot \bar{E} = -i \frac{\omega}{c^2}\frac{\omega_{pi}^2}{\omega_{ci}^2} E_x \Rightarrow \nabla \times \mu_0 J = i \bar{k} \times \hat{x}\left(-i \frac{\omega}{c^2}\frac{\omega_{pi}^2}{\omega_{ci}^2}\right)E_x$$
$$= \frac{k\omega}{c^2}\frac{\omega_{pi}^2}{\omega_{ci}^2} E_x$$
$$= \frac{\omega^2}{c^2}\frac{\omega_{pi}^2}{\omega_{ci}^2}\delta B_Y$$

OR

$$\underbrace{\nabla \times (\nabla \times \delta B_Y)}_{\nabla(\nabla \cdot B) - \nabla^2 \delta B_Y} = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(1 + \frac{\omega_{pi}^2}{\omega_{ci}^2}\right)\delta B_Y$$

$$\hookrightarrow \quad \frac{\partial^2}{\partial z^2}\delta B_T = \frac{1}{V_A^2}\frac{\partial^2 \delta B_Y}{\partial t^2} \qquad \text{QED}$$

#4.19 TO SHOW ALFVÉN WAVES TO BE COMPRESSABLE, WE NEED TO FIND

$$\nabla \cdot nU = n_0 \, i \bar{k} \cdot \bar{V}$$

WHERE $\bar{V} = \bar{E} \times \bar{B}_0 / B_0^2$. THEREFORE, IF

$\bar{k} \cdot \bar{E} \times B_0 = 0$ NOT COMPRESSABLE

$\bar{k} \cdot \bar{E} \times B_0 \neq 0$ COMPRESSABLE

$$\bar{k} \cdot \bar{E} \times \bar{B}_0 \propto \hat{z} \cdot (\bar{k} \times \bar{E}) = \hat{z} \cdot \big(\hat{z} k_x E_y - \hat{y} k_x E_z + \hat{y} k_z E_x - \hat{x} k_z E_y \big)$$
$$= k_x E_y$$
$$= k \sin\theta \, E_y$$

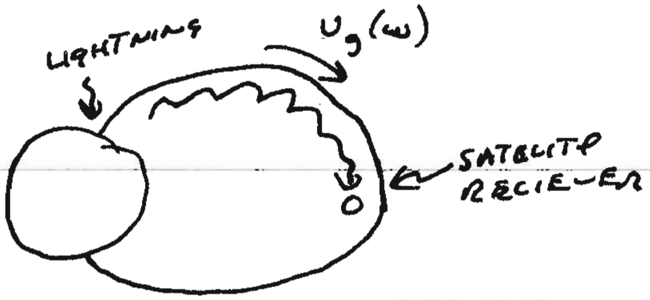
TO FIND THE POLARIZATION, WE WRITE THE EIGENSYSTEM AS

$$\begin{bmatrix} S - n^2\cos^2\theta & 0 & n^2 \sin\theta\cos\theta \\ 0 & S - n^2 & 0 \\ n^2 \sin\theta\cos\theta & 0 & P - n^2 \sin^2\theta \end{bmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0$$

IF $S = n^2$, THEN $E_y \neq 0$ AND THE ALFVEN WAVE IS <u>COMPRESSABLE</u>

IF $S = n^2\cos^2\theta$, THEN $E_y = 0$ (UNLESS $\theta = 0, \pi$) AND $E_x \neq 0$. SHEAR ALFVEN WAVES ARE <u>INCOMPRESSIBLE</u>

Q#5 | THE DELAY BETWEEN TONE ARRIVAL IS



$$\tau(\omega) = \int_{\text{LIGHTNING}}^{\text{REC}} \frac{dl}{v_g(\omega)}$$

$$\Delta\tau \equiv \tau(2\text{kHz}) - \tau(5\text{kHz}) = 2 \text{ sec} \quad (\text{MEASUREMENT})$$

$$\Delta\tau = \int_L^R dl \left(\frac{1}{v_g(2\text{kHz})} - \frac{1}{v_g(5\text{kHz})} \right)$$

BUT $\quad n^2 = \dfrac{\omega_{pe}^2}{\omega\,\omega_{ce}} \Rightarrow \omega = c^2 k^2 \omega_{ce}/\omega_{pe}^2$

$$\frac{\partial\omega}{\partial k} = \frac{2\omega}{k} = 2c\sqrt{\frac{\omega\,\omega_{ce}}{\omega_{pe}^2}}$$

THUS

$$\Delta\tau = \int_L^R \frac{dl}{2c}\sqrt{\frac{\omega_{pe}^2}{\omega_{ce}}}\left(\frac{1}{\sqrt{2\pi\, 2\text{kHz}}} - \frac{1}{\sqrt{2\pi\, 5\text{kHz}}} \right)$$

$$= \int_L^R \frac{dl}{2V_A}\sqrt{\omega_{ce}}\left(\frac{1}{\sqrt{2\pi\, 2\text{kHz}}} - \frac{1}{\sqrt{2\pi\, 5\text{kHz}}} \right) \times \sqrt{\frac{m_e}{m_i}}$$

FOR SIMPLICITY, I WILL USE B AT EQUATOR AND ASSUME $V_A \doteq$ CONSTANT. B AT EQUATOR IS 0.02 GAUSS AT $5.8 R_J$ SO $f_{ce}(B_0) \sim 57.0$ kHz, SO $\times\sqrt{m_e/m_i}$

$$\Delta\tau \sim \frac{\Delta l}{2V_A}\left(\sqrt{\frac{57.0\text{kHz}}{2\text{kHz}}} - \sqrt{\frac{57.0\text{kHz}}{5\text{kHz}}} \right) = \frac{\Delta l}{2V_A} \times 0.05$$

THUS $V_A \sim 0.025 \dfrac{\Delta l}{\Delta\tau}$. TAKE $\Delta l \sim \frac{\pi}{2}\, 5.8 \times 71\times10^6$ m, $V_A \doteq 8\times10^6 \frac{m}{sec}$