10/20/05

APPH 6101 HW #6 SOLUTIONS

#5.3) a) $\displaystyle\int_{-\infty}^{\infty} F\,dv = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{dv/c}{\left(1+(v/c)^2\right)^2}$ LET $x = v/c$, $dx = dv/c$

$$= \frac{2}{\pi}\,\frac{1}{2}\left[\frac{x}{1+x^2} + \tan^{-1}[x]\right]_{-\infty}^{\infty} = 1$$

b) $\displaystyle\int_{-\infty}^{\infty} F\,v^2\,dv = \frac{2}{\pi}c^2\int_{-\infty}^{\infty}\frac{dx\,x^2}{(1+x^2)^2} = c^2$

(INTEGRALS FROM TABLES / MATHEMATICA)

#5.4)

a) $\displaystyle 1 \stackrel{?}{=} \iiint d^3v\,f = \int_0^{2\pi} d\varphi \int_0^{\infty} v_\perp dv_\perp \int_{-\infty}^{\infty} dv_\parallel\, f$

$$= \frac{1}{\sqrt{2\pi}}\int_0^{\infty}\frac{x\,dx\,(x^2/2)^\ell}{\ell!}\int_{-\infty}^{\infty} dY\, e^{-x^2/2}\,e^{-Y^2/2}$$

WITH $x = v_\perp / c_\perp$, $Y = v_\parallel / c_\parallel$

BUT $\displaystyle\int_{-\infty}^{\infty} dY\, e^{-Y^2/2} = \sqrt{2\pi}$

$$\int_0^{\infty} x\,dx\,(x^2/2)^{\ell}\, e^{-x^2/2} = \int_0^{\infty} dz\, z^{\ell} e^{-z} \quad (z = x^2/2)$$
$$= \ell!$$

Q.E.D.

b) $\displaystyle P_\parallel = \iiint d^3v\, m\, v_\parallel^2 f \qquad P_\perp = \iiint d^3v\, m\, v_x^2 f = \iiint d^3v\, m\, v_y^2 f$

SO

$$P_\parallel = \frac{n\,m\,c_\parallel^2}{\sqrt{2\pi}\,\ell!}\int_0^{\infty} x\,dx\,(x^2/2)^\ell \int_{-\infty}^{\infty} dY\, Y^2 e^{-x^2/2}\, e^{-Y^2/2}$$
$$= n\,m\,c_\parallel^2\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dY\, Y^2 e^{-Y^2/2} = n\,m\,c_\parallel^2$$

#5.4 CONT. AND

$$P_\perp = \frac{n m c_\perp^2}{\sqrt{2\pi}\, \ell!} \underbrace{\int_0^{2\pi} d\varphi \frac{\sin^2\varphi}{2\pi}}_{\frac{1}{2}} \underbrace{\int_0^{\infty} x^3 dx \,(x^2/2)^\ell}_{\downarrow} \underbrace{\int_{-\infty}^{\infty} dy}_{\sqrt{2\pi}}\, e^{-x^2/2 - y^2/2}$$

$$\frac{1}{\ell!}\int_0^\infty x^3 dx\,(x^2/2)^\ell e^{-x^2/2} = \frac{2}{\ell!}\int_0^\infty dz\, z^{\ell+1} e^{-z}$$

$$= 2\,\frac{(\ell+1)!}{\ell!} = 2(\ell+1)$$

SO $\quad P_\perp = n m c_\perp^2 \, \frac{1}{2} \times 2(\ell+1)$

$\quad = n m c_\perp^2 (\ell+1) \qquad$ Q.E.D.

#5.5 $\quad \frac{\partial f}{\partial t} + v\cdot\nabla f - \frac{q}{m}\nabla\varphi\cdot\nabla_v f = 0$

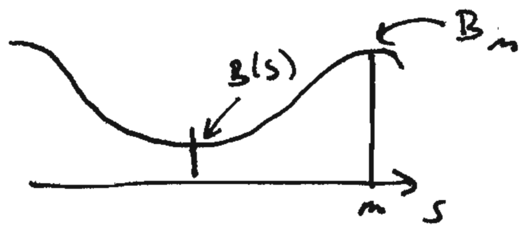
BUT $\frac{\partial f}{\partial t} = 0$; $\quad v\cdot\nabla f = v\cdot\nabla\varphi\left(-\frac{f q}{k T_e}\right)$

AND

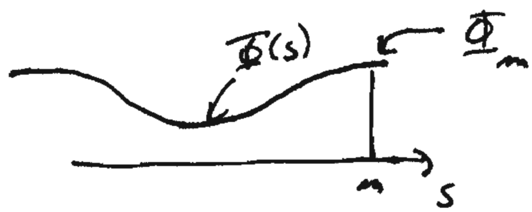
$\nabla\varphi\cdot\nabla_v f = -\nabla\varphi\cdot\left(\frac{m\bar{v}}{kT}\right) f$

SO $\quad 0 - \frac{f q}{k T_0}\,\bar{v}\cdot\nabla\varphi - \frac{q}{m}\left(\frac{m f}{kT} - \bar{v}\cdot\nabla\varphi\right) = 0 \quad$ ✓

#5.7 $\quad B(s)$



EXAMPLE: $\quad \Phi(s)$



5.7 CONT.)

CONSTANTS OF MOTION: $\mu = \frac{1}{2} m v_\perp^2(s) / B(s)$

$W = \mu B(s) + \frac{1}{2} m v_\parallel^2(s) + q\Phi(s)$

a) LOSS-CONE IS DEFINED AS $v_\parallel(s=m) = 0$

THEREFORE

$$w(s) = w(m)$$

$$\mu B(s) + \frac{1}{2} m v_\parallel^2(s) + q\Phi(s) = \mu B_m + q\Phi_m$$

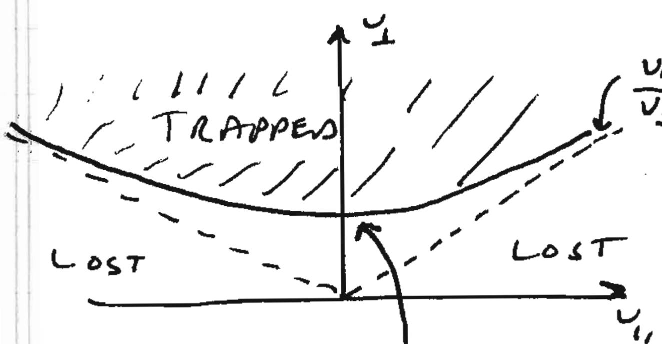
OR

$$v_\parallel^2(s) - \frac{2}{m}\mu\left(B_m - B(s)\right) = \frac{2q}{m}\left(\Phi_m - \Phi(s)\right)$$

$$v_\parallel^2(s) - v_\perp^2(s)\left(\frac{B_m}{B(s)} - 1\right) = \frac{2q}{m}\left(\Phi_m - \Phi(s)\right)$$

b) IF $B_m > B(s)$ AND $\Phi_m > \Phi(s)$, THEN VELOCITY SPACE-BOUNDARY AT s IS

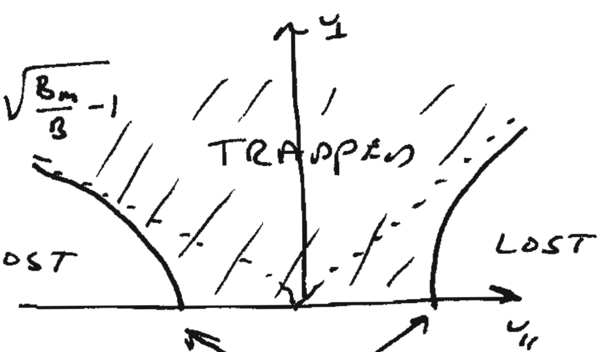
ELECTRONS



$$v_\perp = \sqrt{\frac{\frac{2e}{m}\left(\Phi_m - \Phi(s)\right)}{\left(B_m/B(s) - 1\right)}}$$

$q = -e$

IONS



$$v_\parallel = \pm\sqrt{\frac{2e}{M}\left(\Phi_m - \Phi(s)\right)}$$

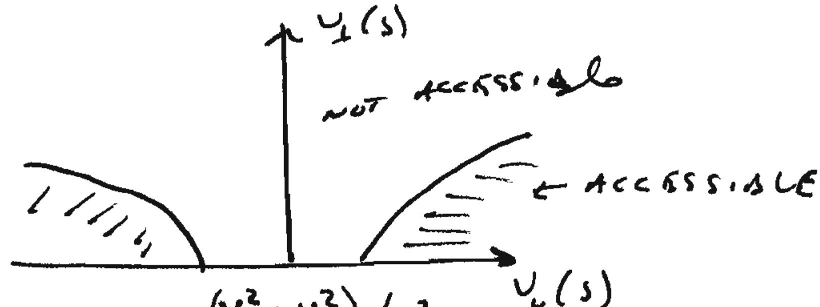
$q = +e$

#5.7 CONT.

c) IF IONS ARE EMITTED AT $s=m$, THEY CAN ONLY ACCESS "LOSS CONE" REGION. IF THEIR INITIAL VELOCITY IS SMALL, THE POTENTIAL ACCELERATES THEM TO $|v_\parallel(s)| = \sqrt{\frac{2e}{m}(\Phi_m - \Phi(s))}$. IF THEY HAVE SOME INITIAL VELOCITY $(v_\perp(m), v_\parallel(s_m))$ THEN THEY WILL HAVE

$$v_\perp(s) = v_\perp(m)\sqrt{\frac{B(s)}{B_m}}$$

$$v_\parallel^2(s) - v_\parallel^2(m) = v_\perp^2(m)\left(1 - \frac{B(s)}{B_m}\right) + \frac{2e}{m}\left(\Phi_m - \Phi(s)\right)$$



d) $f(m) \propto C_2 e^{-(v_\perp^2 + v_\parallel^2)/2c^2}$ where $c^2 = kT/m$

$$f(s) \propto C_2\, e^{-v_\perp^2(s)(B_m/B(s))/(2c^2)}$$

$C_2 \equiv \frac{n}{(\sqrt{2\pi}c)^3}$

$$\times\, e^{-\frac{\left[v_\parallel^2(s) - v_\perp^2(s)\left(\frac{B_m}{B(s)} - 1\right) - \frac{2e}{m}(\Phi_m - \Phi(s))\right]}{2c^2}}$$

or

$$f(s) = C_2\, \mathrm{Exp}\left[-\frac{v_\parallel^2(s)}{2c^2} - \frac{v_\perp^2(s)}{2c^2} + \frac{2e}{m}\left(\Phi_m - \Phi(s)\right)/2c^2\right]$$

ONLY FOR ACCESSIBLE ORBITS