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Lecture 6: Topological Manifolds

1. Local Topological Properties

Exercise 7 of the problem set asked you to understand the notion of path components. We assume that knowledge here. We have done a little work in understanding the notions of connectedness, path connectedness, and compactness. It turns out that even if a space doesn't have these properties globally, they might still hold locally.

Definition. A space X is locally path connected at x if for every neighborhood U of x, there is a path connected neighborhood V of x contained in U. If X is locally path connected at all of its points, then it is said to be locally path connected.

Lemma 1.1. A space X is locally path connected if and only if for every open set V of X, each path component of V is open in X.

Proof. Suppose that X is locally path connected. Let V be an open set in X; let C be a path component of V. If x is a point of V, we can (by definition) choose a path connected neighborhood W of x such that $W \subset V$. Since W is path connected, it must lie entirely within the path component C. Therefore C is open.

Conversely, suppose that the path components of open sets of X are themselves open. Given a point x of X and a neighborhood V of x, let C be the path component of V containing x. Now C is path connected, and since it is open in X by hypothesis, we now have that X is locally path connected at x. Since $x \in X$ was arbitrary, X is locally path connected. $\square$

One similarly defines the local notion of connectedness, and the relevant term is local connectedness. Unfortunately, the definition of *locally compact* is slightly different.

Definition. A space X is said to be locally compact at x if there is some compact subspace C of X that contains a neighborhood of x. If X is locally compact at each of its points, then X is said to be locally compact.

For topological manifolds, the local topological properties tend to be as important as the global properties. And strangely enough, one can deduce global properties of a space if one knows that it satisfies enough local properties. This is illustrated by the last lemma in these notes. It is one example of a *local-to-global question*: can we deduce global properties from local ones? Topological manifolds are the perfect tool for studying this question, because they are global objects defined locally.

2. Topological Manifolds

The prototypical way to describe a topological manifold is to say that it "looks locally like Euclidean space," but what does that really mean? To begin, imagine that you are a very small insect (a flea, perhaps) walking on a very long clothesline. The clothesline might sag in the middle, but you would never notice this, because your vision only goes out to a distance of two inches. Based on what you can see, the clothesline is perfectly straight, and it seemingly has no end. So from your point of view, this clothesline might as well be the real line, $\mathbb{R}$. Perhaps you are also a flea that is afraid to jump. In that case, the clothesline only allows you a single direction of travel - either forward or backward.

Figure 1. Notice the small flea walking on this clothesline

Next, imagine that you are a flea on a giant beach ball (and you are still afraid of jumping). This time you are allowed two directions of travel - you can go either forward or backward, in addition to going either left or right. But just as in the last example, your limited vision would make you unable to tell the difference between this beach ball and a totally flat surface. From your point of view, this beach ball might as well be the Euclidean plane $\mathbb{R}^2$.

These examples capture the spirit of local Euclidean-ness. This term, which we will define shortly, should be imagined from the point of view of a flea. To say that a space is locally Euclidean means that you would be unable to tell the difference between that space and some Euclidean space, from a flea's perspective. The number of degrees of freedom that you have in that space is the *dimension*, and it is an invariant of a locally Euclidean space. The fact that the ground we walk on is locally Euclidean of dimension 2 is almost certainly what led ancient peoples to believe that the world is flat!

Definition. A topological space X is said to be locally Euclidean (of dimension n) if for each point $x \in X$ there is a neighborhood of x that is homeomorphic to an open set of $\mathbb{R}^n$.

Example. Naturally, most "nice" subspaces of a Euclidean space will be locally Euclidean, but that does not mean that every locally Euclidean space must be sitting inside some ambient Euclidean space.

Now we are ready to talk about manifolds. These objects can take many forms, but hindsight has shown that the definition that follows is the most useful way to study these objects. Think back to the two flea examples. Even though an outside observer would know that the clothesline and the beach ball are sitting in our 3-dimensional universe, the flea would not know that. Since the flea is afraid to jump, he can only move around on the surface of these two objects. So to him, these objects are his entire universe. This is exactly the mindset we should adopt when thinking about manifolds. We often draw manifolds as being embedded in Euclidean space, but really these are topological spaces in their own right. They just happen to have the locally Euclidean property, which is what makes them really special.

Definition. Let M be a topological space. We say that M is a topological manifold (of dimension n) if M satisfies the following three properties:

(a) M is Hausdorff
(b) M is second countable
(c) M is locally Euclidean of dimension n.

Notation. Some topologists abbreviate the phrase "Let M be a manifold of dimension n" by writing "Let M^n be a manifold." The superscript n is not part of the name of the manifold, and it is usually only included in the first mention of the manifold.

Let us also fix some notation to describe the locally Euclidean structure of a manifold. We are supposing that for every point p of M we have

(a) an open set $U \subset M$ containing p,
(b) an open set $\tilde{U} \subset \mathbb{R}^n$, and
(c) a homeomorphism $\varphi\colon U \to \tilde{U}$.

We call the pair (U, φ) a coordinate chart (or just a chart). By definition of a topological manifold, every point $p \in M$ is contained in the domain of some chart. If $\varphi(p) = 0$, then we say that the chart is centered at p. The set U in a chart is called the coordinate domain, and the map φ is called a coordinate map. Finally, the component functions of φ, defined by $\varphi(p) = (x^1(p), ..., x^n(p))$ are called local coordinates on U. One usually thinks of local coordinates as actually defining coordinates on the manifold. More precisely, once we choose a chart (U, φ) on M, the coordinate map $\varphi\colon U \to \tilde{U}$ should be thought of as giving an identification between

U and $\tilde{U}$. In this way, we can think of U both as an open subset of M and (while working in this chart) as an open subset of $\mathbb{R}^n$. One visualizes this identification by thinking of a "grid" drawn on U, representing the inverse images of the coordinate gridlines under φ. See figure 2.

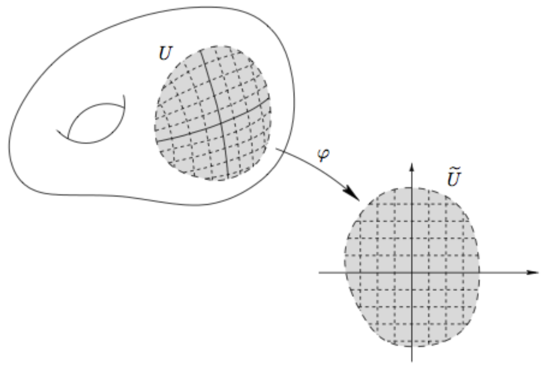

FIGURE 2. A coordinate grid

Example. Graphs of continuous functions are topological manifolds. Let $D \subset \mathbb{R}^n$ be an open set, and let $F\colon D \to \mathbb{R}^k$ be a continuous function. The *graph* of F is a subset of $\mathbb{R}^n \times \mathbb{R}^k$ given by

$$\Gamma(F) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^k \colon x \in D \text{ and } y = F(x)\},$$

with the subspace topology. Let $\pi\colon \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ be the map that projects onto the first factor, $(u, v) \mapsto u$. Let $\varphi_F\colon \Gamma(F) \to \mathbb{D}$ denote the restriction of π to the graph $\Gamma(F)$. Then φ_F will be continuous, because it is the restriction of a continuous projection map. The inverse map is given by

$$(\varphi_F)^{-1}(x) = (x, F(x)),$$

and this is obviously continuous. Therefore φ_F is a homeomorphism. So we have shown that $\Gamma(F) \cong D \subset \mathbb{R}^n$, and this means that $\Gamma(F)$ is locally Euclidean. It is Hausdorff and second countable, because it is a topological subspace of $\mathbb{R}^{n+k}$. Therefore it is a topological n-manifold.

The coordinate chart we exhibited works for all of $\Gamma(F)$. We call such a thing a global coordinate chart, because it works everywhere.

Example. The unit circle $S^1 = \{(x_1, x_2) \in \mathbb{R}^2 \colon x_1^2 + x_2^2 = 1\}$ is a topological manifold of dimension 1, when topologized using the subspace topology. This space is Hausdorff and second countable, because it is a subspace of $\mathbb{R}^2$. We need only show that it is locally Euclidean. Let U_i^+ be the subset of S^1 where the i^{th} coordinate is positive, and let U_i^- be the subset where the i^{th} coordinate is negative. So we have

$$U_1^+ = \{(x_1, x_2) \in \mathbb{S}^1 \colon x_1 > 0\}, \qquad U_2^+ = \{(x_1, x_2) \in \mathbb{S}^1 \colon x_2 > 0\},$$

$$U_1^- = \{(x_1, x_2) \in \mathbb{S}^1 \colon x_1 < 0\}, \qquad U_2^- = \{(x_1, x_2) \in \mathbb{S}^1 \colon x_2 < 0\}.$$

Each $U_i^\pm$ is a graph over $(-1, 1)$ of a semicircle. By the previous example, these graphs are locally Euclidean. We can also write out the coordinate function explicitly.

$$\varphi_i^\pm(x_1, x_2) = (x_1, \hat{x}_i, x_2)$$

Lemma 2.1. A topological manifold M has a countable basis of open coordinate balls, the closure of each of which is a compact set. Therefore, we may apply the following descriptors to any topological manifold.

(a) It is *locally connected*;
(b) it is *locally path connected*;
(c) it is *locally compact*.

Finally, in a locally path-connected space, the connected components and the path connected components coincide.

Proof. The statements (a) - (c) are just definitions. Verifying these is an exercise, and requires that M be T_2 (which it is). We will only show the existence of such a basis. We know that M is second-countable, so any open cover has a countable sub-cover. Therefore M can be covered by countably many coordinate charts. If we can find a countable basis of open coordinate balls for a single coordinate chart of M, then we can do the same for each coordinate chart, and the collection of all the balls in all the charts will still be a countable basis for M. (A countable union of countable sets is countable). So without a loss of generality, we may assume that M can be covered by a single coordinate chart. But if this is the case, then M is homeomorphic to an open subset of $\mathbb{R}^n$. Let us say that $\varphi \colon M \to \tilde{U} \subset \mathbb{R}^n$ is this homeomorphism. We know the standard topology of $\mathbb{R}^n$ is generated by all open balls. We reduce this to a countable basis by allowing only balls with rational radii, centered at points with rational coordinates. This collection of balls covers all of $\mathbb{R}^n$ and so the inverse image $\varphi^{-1}(B)$ of each ball B that meets $\tilde{U}$ will be a countable basis for M.

According to Lemma 1.1, M is locally path connected if and only if for every open set V of M, the path components of V are open sets of M. Thus the path components give a partition of M into disjoint open sets. So any open, connected subspace of M is path connected. Each connected component of M is connected and locally path connected, hence path connected. We have shown that the connected components and the path connected components coincide. □

Remark. The final sentence in the statement of this lemma shows why Hall's book substitutes the definition for *path connected* in place of the definition for *connected*