

A Two-Dimensional Logic for Diagonalization and the A Priori[‡]

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A Introduction

Two-dimensional semantics, which can represent a distinction between apriority and necessity, has wielded considerable influence in the philosophy of language. In this paper, I axiomatize the logic of the \dagger operator of [Stalnaker \(1978\)](#) in the context of two-dimensional modal logic. This operator traces its ancestry through [Segerberg \(1973\)](#) and [Åqvist \(1973\)](#).

In two-dimensional modal logic, the truth of a formula ϕ in a model \mathcal{M} is relativized to two parameters: a parameter representing the *world-as-actual*, which in our semantic notation (as in $(@)$, (\dagger) , and (\blacksquare) , below) appears first, and a parameter representing the *world of evaluation*, which we write second.¹ My exposition leans on Stalnaker’s notion of *diagonalization*, and his characterization of a priori/necessary (A/\blacksquare) distinction as the contrast between the truth of $\blacksquare \dagger \phi$ and the truth of $\blacksquare \phi$. The Kripke models used for this result are shown to be equivalent to Stalnaker’s matrix models, which are well known in philosophy of language.

I call this proof system B2D, for “basic two-dimensionalism”. The object language contains both “actually” $(@)$ and the \dagger operator.

$(@)$ $\mathcal{M}, y, x \vDash @\phi$ iff $\mathcal{M}, y, y \vDash \phi$

(\dagger) $\mathcal{M}, y, x \vDash \dagger\phi$ iff $\mathcal{M}, x, x \vDash \phi$

The language also contains a global S5 modality \blacksquare common in discussions of two-dimensional semantics ([Crossley & Humberstone, 1977](#); [Davies & Humberstone, 1980](#); [Fritz, 2013, 2014](#)):

(\blacksquare) $\mathcal{M}, y, x \vDash \blacksquare\phi$ iff for all x' : $\mathcal{M}, y, x' \vDash \phi$

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¹There is considerable variation in terminology for the first parameter, and indeed whether it should be thought of as a world, or something else (usually richer). Prominently, this parameter corresponds to the *world of the context* in [Kaplan \(1989\)](#), and to the *scenario* parameter in [Chalmers \(2004\)](#).

The focus on \dagger , and the gloss on apriority as $\blacksquare\dagger$, marks a contrast with the classic literature on two-dimensional modal logic, which is centered on $@$.²³

Despite this traditional focus, the dagger maintains its unique interest. It is useful to have separate representations of \dagger and \blacksquare , for example, to model the distinctive maneuvers of “diagonalizers” (Stalnaker, 1978) and “supervaluers” (Thomason, 1970) in cases where the indeterminacy of the open future affects both the content and the truth-conditions of future-directed discourse (MacFarlane, 2009; Fusco, 2019). In a quite different context, the notion of diagonalization—the interpretation of ϕ as $\dagger\phi$ —been used in formal epistemology to model subtleties of Bayesian learning. Stalnaker (2010)’s discussion of the Sleeping Beauty problem (Elga, 2000), for example, suggests that there is a proposition Beauty learns when she wakes up after the coin-toss experiment which is distinct from a superficially indistinguishable proposition she knew before—the key difference being that the first proposition is the diagonalization of the second.⁴ These applications are conceptually at home in a framework which treats \dagger on a par with A . Other \dagger -centric work in the literature includes a brief discussion of \dagger in Ch. 10 of Sider (2010), under the name ‘ \times ’, and the discussion in Cresswell (2012), under the name ‘*Ref*’. Closer to the present study, a completeness theorem for tableau systems featuring $@$, A , and \dagger (under the name ‘ \otimes ’) appears in Lampert (2018).

In the final section of the paper, I discuss the relationship between the proof system presented here and *Åqvist logics* (Åqvist, 1973; Segerberg, 1973, pg. 77), which intuitively privilege the space of *worlds* over the space of *world-pairs*—and hence the space of *propositions* over the space of general *dipositions* studied here.⁵

A.1 Two Dimensionalism via \dagger : an outline

Stalnaker’s classic discussion in “Assertion” introduces two-dimensional semantics by injecting uncertainty into a situation where speakers express thoughts using context-sensitive language.

Let me give a simple example: I said *you are a fool* to O’Leary. O’Leary is a fool, so what I said was true, although O’Leary does not think so. Now Daniels, who is no fool and who knows it, was standing near by, and he thought I was talking to him. So both O’Leary and Daniels thought I said something false: O’Leary understood what I said, but disagrees with me about the facts; Daniels, on the other hand, agrees with me about the fact ...but misunderstood what I said. Just to fill out the example, let me add that O’Leary believes falsely that Daniels is a fool. Now compare the possible worlds i, j , and k . i is the world as it is ... j is the world that O’Leary thinks we are in; and k is the world Daniels thinks we are in...the following TWO-DIMENSIONAL matrix also represents the [first and] second way[s] that the truth-value of my utterance is a function of the facts.

²For other work centered around $@$, see, *inter alia*, Crossley & Humberstone (1977); Gregory (2001); Blackburn & Marx (2002); Hazen et al. (2013); Fritz (2013, 2014).

³It is worth noting, in this respect, that apriority (A) can also be expressed via the compound “fixedly actually” ($\mathcal{F}@$) operator discussed by Davies & Humberstone (1980). The semantic entry for \mathcal{F} is:

$$(\mathcal{F}) \quad \mathcal{M}, y, x \vDash \mathcal{F}\phi \text{ iff for all } y': \mathcal{M}, y', x \vDash \phi$$

⁴See the exposition in Magidor (2010). Similar arguments play a role in *screening-off* arguments in decision theory (Ahmed, 2014, Ch. 4.1, Fusco, 2018).

⁵The term *diposition* comes from Humberstone (1981), whereas *propositional concept* is used by Stalnaker *op. cit.*.

	i	j	k			
i	T	F	T	$i, i \models p$	$j, i \models p$	$k, i \not\models p$
j	T	F	T	$i, j \not\models p$	$j, j \not\models p$	$k, j \models p$
k	F	T	F	$i, k \models p$	$j, k \models p$	$k, k \not\models p$

$p = \ulcorner \text{You are a fool} \urcorner$

Figure 1: A two-dimensional matrix (figure from Stalnaker (1978)).

In Stalnaker’s 3×3 diagram, the vertical axis represents different values of the world-as-actual parameter, while the horizontal axis represents different values of the world of evaluation parameter. In this particular example, the vertical axis disambiguates different resolutions of the context-sensitive expression “you”, and the horizontal axis shows worlds i, j , and k in their role as arguments to the relevant propositions expressed. Stalnaker points out that the diagram represents the speaker and O’Leary as agreeing on what *proposition* was expressed by the utterance (since the same row of truth-values, T, F, T, appears to the right of both i and j), even though O’Leary believes what was said was false.⁶ That is, the proposition O’Leary thinks was expressed ($i \rightarrow \text{T}, j \rightarrow \text{F}, k \rightarrow \text{T}$) is false in the world he thinks he is in (viz., j). The fact that the same *column* appears under i and k represents that the speaker and *Daniels* agree on who is a fool and who isn’t: both the-world-according-to-the-speaker (viz., i) and the-world-according-to-Daniels (viz., k) agree, for each context-insensitive disambiguation of the “you” in *you are a fool* (respectively: O’Leary in worlds i and j , and Daniels in world k), on whether the proposition expressed by that disambiguated claim is true or false.

The “you are a fool” example involves a misunderstanding of an utterance (on Daniels’s part). But sometimes a failure to know how context sensitivity is to be resolved is recognized by all parties. When several candidate worlds are thus ineliminable, a sentence ϕ can be rationally (re)interpreted as $\dagger\phi$: as Stalnaker puts it, the \dagger takes the diagonal of a matrix and “projects” it into the horizontal (*op. cit.*, pg. 82). This is *diagonalization*.

	i	j	k		
i	T	F	T		
j	T	F	T		
k	F	T	F		

2D matrix for $\ulcorner \text{you are a fool} \urcorner$

	i	j	k
i	T	F	F
j	T	F	F
k	T	F	F

2D matrix for $\ulcorner \dagger (\text{you are a fool}) \urcorner$

Figure 2: Diagonalization.

A diagonalized matrix satisfies the condition that the same proposition—the same “horizontal” function from worlds to truth-values—is expressed relative to every candidate for actuality (*op. cit.*, pg. 88).⁷ This is a precondition of identifying the *context change potential*, or update of the conversational common ground, associated with the information intuitively carried by an assertion of ϕ (Heim, 1982; Stalnaker, 2002). The terminology of

⁶As is common in the literature, I will move freely between characterizing propositions as sets of possible worlds and as characteristic functions of those sets.

⁷Hawthorne & Magidor (2009) call this feature “Uniformity”, noting that diagonalization does not restore Uniformity in non-S5 frameworks (p. 382). Stalnaker himself relates the principle to *Tractatus* Proposition 2.0211 (Wittgenstein, 1974).

the a priori and the a posteriori applies because $\blacksquare \dagger \phi$ represents update on the proposition expressed by $\dagger \phi$, which—unlike the proposition expressed by ϕ —is something interlocutors are in a position to know prior to working out exactly which world they are in. Stalnaker goes on to apply this treatment to the contingent a priori sentence “This bar [the meter bar] is one meter long” (*op. cit.*, pp. 83-84; see [Kripke, 1980](#); [Wittgenstein, 1953](#)); the proposition expressed by the sentence might have been false, but since the sentence cannot be falsely uttered, it does not rule out any worlds as candidates for actuality.

Here, we investigate the logic of the two-dimensional language of [Crossley & Humberstone \(1977\)](#), augmented by \dagger . A completeness result, with respect to a class of matrix models for the proof system introduced in the next section, follows closely the strategy of [Fritz \(2014\)](#), §2.

B The logic B2D

Definition 1. Language:

$\mathcal{L} := At \mid (\phi \wedge \phi) \mid \neg\phi \mid \dagger\phi \mid @\phi \mid \blacksquare\phi$

$\blacklozenge, \vee, \rightarrow$ and \equiv defined as usual; $A\phi$ (*it is a priori that* ϕ) defined as $\blacksquare \dagger \phi$.

Definition 2. Let B2D be the proof system axiomatized by Modus Ponens and Uniform Substitution; all Boolean tautologies; the K Axiom and Necessitation for each primitive modal operator; and the following axioms:

$$\begin{array}{ll}
(T_{\blacksquare}) \blacksquare p \rightarrow p & (F_{\dagger}) \dagger p \equiv \neg \dagger \neg p \\
(5_{\blacksquare}) \blacklozenge p \rightarrow \blacksquare \blacklozenge p & (X_{\rightarrow}) \dagger(p \rightarrow @p) \\
(@5_{\blacklozenge}) \blacklozenge @p \rightarrow @p & (Y_{\rightarrow}) @ (p \rightarrow \dagger p) \\
(G) \blacksquare p \rightarrow @p & (4_A) \blacksquare \dagger p \rightarrow \blacksquare \dagger \blacksquare \dagger p \\
(F_{@}) @p \equiv \neg @ \neg p & (5_A) \blacklozenge \dagger p \rightarrow \blacksquare \dagger \blacklozenge \dagger p
\end{array}$$

Notation: $\vdash_{B2D} \phi$ holds if ϕ belongs to the smallest set of formulas that contains all the axioms and is closed under the rules of inference.

While \mathcal{L} augments the language of [Crossley & Humberstone \(1977\)](#) and [Fritz \(2013, 2014\)](#), it is a fragment of the larger language of [Seegerberg \(1973\)](#). First we prove some Facts in B2D that are familiar from this literature.

Fact 1. $\vdash_{B2D} \dagger(p \equiv @p)$

Proof. The right-left direction of the embedded biconditional is an axiom.

We reason with the following equivalences:

1. $\dagger(\neg p \rightarrow @\neg p)$ instance of (X_{\rightarrow})
2. $\dagger(\neg @ \neg p \rightarrow p)$ from (1), Contraposition
3. $\dagger(@p \rightarrow p)$ from (2), $(F_{@})$, DNE
4. $\dagger(@p \equiv p)$ from (3), (X_{\rightarrow}) , Necessitation and K Axiom for \dagger

Call this theorem (X) and call the theorem $\dagger(@p \rightarrow p)$, which was proved at Step 3 above, (X_{\leftarrow}) . \square

Fact 2. $\vdash_{B2D} @ (p \equiv \dagger p)$

Proof. The right-left direction is an axiom. The left-right direction is immediate from a proof similar to the proof of Fact 1. Call this theorem (Y), and call the theorem $@(\dagger p \rightarrow p)$, which was proved at Step 3 in the analogous proof to the proof of Fact 1 above, (Y_{\leftarrow}). \square

Fact 3. $\vdash_{B2D} @p \rightarrow \blacksquare @p$

Proof. From ($@5_{\blacklozenge}$) and ($F_{@}$). Call this theorem ($@5_{\blacksquare}$) (Crossley & Humberstone, 1977, pg. 14).⁸ \square

Fact 4. $\vdash_{B2D} \dagger \blacklozenge \dagger p \equiv \blacklozenge \dagger p$

Proof. The left-right direction is immediate from (4_A) and (T_{\blacksquare}). The right-left direction is immediate from (5_A) and (T_{\blacksquare}). Call this biconditional ($\dagger \blacklozenge \dagger$), or the *columnarity* of R_{\blacksquare} . \square

Fact 5. $\vdash_{B2D} \dagger \dagger p \equiv \dagger p$

Proof. We reason with the following equivalences:

1. $\dagger @p \equiv \dagger p$ from Fact 1, K axiom for \dagger
2. $@ \dagger p \equiv @p$ from Fact 2, K axiom for $@$
3. $\dagger @ \dagger p \equiv \dagger \dagger p$ from (1) (substituting p with $\dagger p$)
4. $\dagger @ \dagger p \equiv \dagger @p$ from (2), Necessitation with \dagger
5. $\dagger \dagger p \equiv \dagger p$ from (1), (3), (4), Transitivity

Call this (Red_{\dagger}). \square

Fact 6. $\vdash_{B2D} @@p \equiv @p$

Proof. This can be proven analogously to the proof of Fact 5. \square

Call this ($\text{Red}_{@}$).

Fact 7. $\vdash_{B2D} \blacksquare \dagger p \rightarrow @p$

Proof. From (G) and (Y) (viz., Fact 2). By instantiating ' $\dagger p$ ' for p , we get $\blacksquare \dagger p \rightarrow @ \dagger p$ from (G). By distributing the $@$ operator over (Y), we get $@ \dagger p \equiv @p$. Hence $\blacksquare \dagger p \rightarrow @p$ by transitivity of \rightarrow .

Call this ($A_{@}$). \square

Summarizing these facts:

⁸Because Crossley & Humberstone use ' A ' for "actually" and ' \square ' for our \blacksquare , their version is called ' $(A5_{\square})$ '.

(X) $\dagger(p \equiv @p)$	$(\blacklozenge \dagger \blacklozenge)$ $\dagger \blacklozenge \dagger p \equiv \blacklozenge \dagger p$
(X_{\leftarrow}) $\dagger(@p \rightarrow p)$	(Red_{\dagger}) $\dagger \dagger p \equiv \dagger p$
(Y) $@(p \equiv \dagger p)$	$(\text{Red}_{@})$ $@@p \equiv @p$
(Y_{\leftarrow}) $@(\dagger p \rightarrow p)$	$(A@)$ $\blacksquare \dagger p \rightarrow @p$
$(@5_{\blacksquare})$ $@p \rightarrow \blacksquare @p$	

B.1 The class Fr_{B2D}

Definition 3. A **Kripke frame** is a tuple $\mathcal{K} = \langle W, \{R_{\nabla_i} : i \in I\} \rangle$ where W is a set of points and $R_{\nabla_i}, i \in I$ are binary relations on W .

Fact 8. The formulas in B2D have the following global first-order correspondents in Kripke frames (where ‘ $uR_{\blacksquare} \circ R_{\dagger}v$ ’ abbreviates relation $\exists z : uR_{\blacksquare}zR_{\dagger}v$):

(T_{\blacksquare}) R_{\blacksquare} is reflexive	$(F_{@})$ $R_{@}$ is a function
(5_{\blacksquare}) R_{\blacksquare} is Euclidean	(X_{\rightarrow}) $\forall w, v, u : (wR_{\dagger}v \wedge vR_{@}u \rightarrow v = u)$
$(@5_{\blacklozenge})$ $\forall w, v, u : (wR_{\blacksquare}u \wedge wR_{@}v) \rightarrow uR_{@}v$	(Y_{\rightarrow}) $\forall w, v, u : (wR_{@}v \wedge vR_{\dagger}u \rightarrow v = u)$
(G) $R_{@}$ is a subrelation of R_{\blacksquare}	(4_A) $R_{\blacksquare} \circ R_{\dagger}$ is transitive
(F_{\dagger}) R_{\dagger} is a function	(5_A) $\forall x, y, a, b[(xR_{\blacksquare}y \wedge xR_{\blacksquare}a \wedge aR_{\dagger}b) \rightarrow \exists c(bR_{\blacksquare}c \wedge \forall d(cR_{\dagger}d \rightarrow yR_{\dagger}d))]$

Fact 9. Additionally, theorems (X_{\leftarrow}) , (X) , (Y_{\leftarrow}) , (Y) , and $(@5_{\blacklozenge})$ (c.f. Facts 1-3) have the following local first-order correspondents in Kripke frames:

(X_{\leftarrow}) within $\text{Img}(R_{\dagger})$, $R_{@}$ is reflexive	(X) within $\text{Img}(R_{\dagger})$, $R_{@}$ is the identity relation
(Y_{\leftarrow}) within $\text{Img}(R_{@})$, R_{\dagger} is reflexive	(Y) within $\text{Img}(R_{@})$, R_{\dagger} is the identity relation
$(@5_{\blacksquare})$ $R_{\blacksquare} \circ R_{@} \subseteq R_{@}$	

Definition 4. A **basic matrix frame** \mathcal{M} is a Kripke frame $\langle W^X, R_{\blacksquare}^X, R_{@}^X, R_{\dagger}^X \rangle$, where $W^X = X \times X$ for some nonempty set X and:

- $\langle y, x \rangle R_{\blacksquare}^X \langle y', x' \rangle$ iff $y = y'$
- $\langle y, x \rangle R_{@}^X \langle y', x' \rangle$ iff $y' = x' = y$.
- $\langle y, x \rangle R_{\dagger}^X \langle y', x' \rangle$ iff $y' = x' = x$.

The guiding spirit of the completeness proof for the class of basic matrix frames comes from [Fritz \(2014, §1-2\)](#). Following that proof, we identify three types of structure, $\text{M} \subsetneq \text{R} \subsetneq \text{Fr}_{\text{B2D}}$.

1. Fr_{B2D} , the class of Kripke frames on which B2D is valid;
2. R , an intermediate class of ‘‘B-Restall frames’’⁹;

⁹So-called because of their similarity to the frames in [Restall \(2012\)](#).

3. M (the target class), the class of basic matrix frames.

Fr_{B2D} is the class of Kripke frames for which soundness and completeness for the B2D axioms is immediate via Sahlqvist’s Completeness Theorem (Blackburn et al., 2002, Ch. 4). The second class, R , is the class of point-generated subframes of Fr_{B2D} . The smallest class, M , is the class of basic matrix frames, identical to those used by Stalnaker in his discussion of the “you are a fool” and “this bar is one meter long” cases.

We seek to show that these classes are modally equivalent ($\text{Fr}_{\text{B2D}} \rightsquigarrow R \rightsquigarrow M$): that is, that the same sets of formulas are satisfiable in each. To do so, we use two invariance results for modal models: (i) $\text{Fr}_{\text{B2D}} \rightsquigarrow R$ because modal satisfaction is invariant under generated submodels; and (ii) $R \rightsquigarrow M$ because modal satisfaction is invariant under bounded morphisms.

$$\text{Fr}_{\text{B2D}} \xrightarrow{\text{gen. submodel}} R \xrightarrow{\text{bounded morphism}} M$$

Notation: for a relation R_{∇} that is a function, I will use “ $f_{\nabla}(x)$ ” for the unique y s.t. $xR_{\nabla}y$. Let $D = \text{Img}(R_{\textcircled{a}})$. I will use d_1, d_2 , etc. for arbitrary worlds in D .

Now for some additional frame conditions, imposed by the axiom schemata on $\mathcal{F} \in \text{Fr}_{\text{B2D}}$:

Fact 10 (Alternative characterization of diagonal points). $\text{Img}(R_{\textcircled{a}}) = \text{Img}(R_{\dagger})$.

Proof. Let $u \in \text{Img}(R_{\textcircled{a}})$. Hence by (Y_{\leftarrow}) , $u \in \text{Img}(R_{\dagger})$. Conversely, let $u \in \text{Img}(R_{\dagger})$. Hence by (X_{\leftarrow}) , $u \in \text{Img}(R_{\textcircled{a}})$. \square

Fact 11. $R_{\blacksquare} \circ R_{\dagger}$ is Euclidean.

Proof. For brevity, let $R_A := R_{\blacksquare} \circ R_{\dagger}$. Suppose xR_Ay and xR_Az . Thus, there is a y' and z' such that $xR_{\blacksquare}y'$ and $y'R_{\dagger}y$, and also $xR_{\blacksquare}z'$ and $z'R_{\dagger}z$. Since $xR_{\blacksquare}y'$ and $xR_{\blacksquare}z'$ and $z'R_{\dagger}z$, by (5_A) , there is a z'' such that $zR_{\blacksquare}z''$ and for all u , if $z''R_{\dagger}u$, then $y'R_{\dagger}u$. Now, within $\text{Img}(R_{\dagger})$, R_{\dagger} is the identity relation (by (X) and Fact 10.) Hence, if $y'R_{\dagger}u$, then $u = y$. Therefore, if $z''R_{\dagger}u$, then $u = y$. Since R_{\dagger} is a total function, that means $z''R_{\dagger}y$. Hence, we have $zR_{\blacksquare}z''$ and $z''R_{\dagger}y$. So, zR_Ay . \square

Fact 12 (Square Completion (Symmetry) in $\text{Img}(R_{\textcircled{a}})$). $\forall d_1, d_2 \in \text{Img}(R_{\textcircled{a}})$: if $d_1R_{\blacksquare} \circ R_{\dagger}d_2$, then $d_2R_{\blacksquare} \circ R_{\dagger}d_1$.

Proof. Suppose $d_1 \in \text{Img}(R_{\textcircled{a}})$. Then $\exists z : d_1R_{\blacksquare}zR_{\dagger}d_1$: R_{\blacksquare} is reflexive and R_{\dagger} is reflexive within $\text{Img}(R_{\textcircled{a}})$. By assumption, $\exists z : d_1R_{\blacksquare}zR_{\dagger}d_2$. It follows by (5_A) , the Euclideanness of $R_{\blacksquare} \circ R_{\dagger}$, that $\exists z' d_2R_{\blacksquare}z'R_{\dagger}d_1$. \square

Fact 13 ($R_{\blacksquare} \circ R_{\dagger}$ an equivalence relation on $\text{Img}(R_{\textcircled{a}})$). $R_{\blacksquare} \circ R_{\dagger}$ an equivalence relation on $\text{Img}(R_{\textcircled{a}})$.

Proof. This follows from the fact that within $\text{Img}(R_{\textcircled{a}})$, $R_{\blacksquare} \circ R_{\dagger}$ is symmetric (by Fact 12), transitive, and reflexive. \square

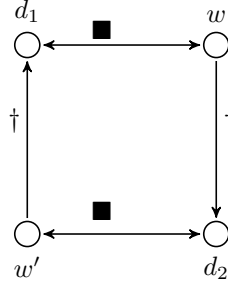


Figure 3: Facts 12 (Square completion) and 14 (Mirror points).

Fact 14 ($A\ddagger$ Mirror Points). Suppose $wR_{\textcircled{a}}d_1$ and $wR_{\ddagger}d_2$. Then $\exists w'$ s.t. $w'R_{\textcircled{a}}d_2$ and $w'R_{\ddagger}d_1$.

Proof. Suppose $wR_{\textcircled{a}}d_1$ and $wR_{\ddagger}d_2$. It follows (from (X_{\leftarrow})) that both $d_1, d_2 \in \text{Img}(R_{\textcircled{a}})$. Because $wR_{\textcircled{a}}d_1$, $wR_{\blacksquare}d_1$ and so $d_1R_{\blacksquare}w$ by symmetry of R_{\blacksquare} . Hence $\exists z : d_1R_{\blacksquare}zR_{\ddagger}d_2$ as witnessed by w . By Fact 12, $\exists z : d_2R_{\blacksquare}zR_{\ddagger}d_1$. Let w' be the z such that $d_2R_{\blacksquare}z \wedge zR_{\ddagger}d_1$. Since $w'R_{\blacksquare}d_2$ and $d_2R_{\textcircled{a}}d_2$, by $(\textcircled{a}5_{\blacklozenge})$ $w'R_{\textcircled{a}}d_2$. Hence $w'R_{\textcircled{a}}d_2$ and $w'R_{\ddagger}d_1$. \square

Theorem 1. B2D is sound and strongly complete with respect to Fr_{B2D} , the class of B2D-frames.

Proof. By the Sahlqvist Completeness Theorem (Blackburn et al., 2002, Ch. 4). \square

B.2 The class R

We seek to show that this class is modally equivalent to the class of B-Restall frames—that a set of formulas is satisfiable on Fr_{B2D} iff it is satisfiable on R, the class of B-Restall Frames. To do so, it will be helpful to prove a Lemma concerning the point-generated universe.

Notation: in general, we write $R[Y]$ for the image of a set Y under R .

Lemma 1. For any point-generated subframe $\mathcal{F}'_w = \langle W', R'_{\blacksquare}, R'_{\ddagger}, R'_{\textcircled{a}} \rangle \in \text{Fr}_{\text{B2D}}$: $W' = R_{\blacksquare}[R_{\ddagger}[R_{\textcircled{a}}[\{w\}]]]$.

Proof. We let $X := R_{\blacksquare}[R_{\ddagger}[R_{\textcircled{a}}[\{w\}]]]$. We show (1) $w \in X$ and (2) that for each operator ∇ that $R_{\nabla}[X] \subseteq X$.

(1) Let $u = f_{\textcircled{a}}(w)$. By (G) , $wR_{\blacksquare}u$. By (Y) , $uR_{\ddagger}u$. By symmetry of R_{\blacksquare} , $uR_{\blacksquare}w$. Hence via u , it follows that $w \in X$.

(2- \blacksquare). Since R_{\blacksquare} is transitive and reflexive, $R_{\blacksquare}[X] = X$.

(2- \ddagger). Let $v \in R_{\ddagger}[X]$. Now $R_{\ddagger}[X] = R_{\ddagger}[R_{\blacksquare}[R_{\ddagger}[R_{\textcircled{a}}[\{w\}]]]]$, by the definition of X . By (4_A) , $R_{\ddagger}[R_{\blacksquare}[R_{\ddagger}[R_{\textcircled{a}}[\{w\}]]]] = R_{\ddagger}[R_{\blacksquare}[\{w\}]]$. Hence it suffices to show that if $v \in R_{\ddagger}[R_{\blacksquare}[\{w\}]]$, then $v \in R_{\blacksquare}[R_{\ddagger}[R_{\textcircled{a}}[\{w\}]]]$. This follows immediately from (T_{\blacksquare}) : viz., from the fact that $vR_{\blacksquare}v$.

(2- \textcircled{a}). Let $v \in R_{\textcircled{a}}[X]$. By Fact 7, $v \in R_{\ddagger}[R_{\blacksquare}[X]]$ and so $v \in R_{\ddagger}[R_{\blacksquare}[R_{\textcircled{a}}[R_{\ddagger}[R_{\textcircled{a}}[\{w\}]]]]]$. By Transitivity of R_{\blacksquare} , $v \in R_{\ddagger}[R_{\blacksquare}[R_{\ddagger}[R_{\textcircled{a}}[\{w\}]]]]]$. By (4_A) , $v \in R_{\ddagger}[R_{\blacksquare}[\{w\}]]$. From (T_{\blacksquare}) , we again conclude that $v \in R_{\blacksquare}[R_{\ddagger}[R_{\textcircled{a}}[\{w\}]]]$. \square

Definition 5. A **B-Restall Frame** is a frame $\mathcal{R} = \langle W, R_{\blacksquare}, R_{\circledast}, R_{\dagger} \rangle$ satisfying three conditions:

1. R_{\blacksquare} is an equivalence relation
2. R_{\circledast} is a function such that
 - (i) $wR_{\circledast}v \rightarrow wR_{\blacksquare}v$
 - (ii) R_{\circledast} maps any two R_{\blacksquare} -related worlds to the same point
3. R_{\dagger} is a function such that
 - (i) for any w : $R_{\dagger}[R_{\blacksquare}[\{w\}]] = \text{Img}(R_{\circledast})$
 - (ii) R_{\dagger} is reflexive over $\text{Img}(R_{\circledast})$

Let \mathbf{R} be the class of B-Restall frames. Let D be the set $\text{Img}(R_{\circledast})$ in a B-Restall frame.

Theorem 2. $\text{Fr}_{\mathbf{B2D}} \leftrightarrow \mathbf{R}$.

Theorem 2 is an consequence of the Lemmas 2 and 3, below. By Lemma 2, any formula falsifiable in $\mathcal{R} \in \mathbf{R}$ is falsifiable in some $\mathcal{F} \in \text{Fr}_{\mathbf{B2D}}$. By Lemma 3, any formula falsifiable in a point-generated subframe $\mathcal{F}_w \in \text{Fr}_{\mathbf{B2D}}$ is falsifiable in some $\mathcal{R} \in \mathbf{R}$.

Because modal satisfaction is invariant under point-generated submodels (Blackburn et al., 2002, Proposition 2.6.), the formulas falsifiable in $\mathcal{R} \in \mathbf{R}$ are *exactly* the formulas falsifiable in $\mathcal{F} \in \text{Fr}_{\mathbf{B2D}}$.

Lemma 2. $\mathbf{R} \subseteq \text{Fr}_{\mathbf{B2D}}$.

Proof. To verify this, it suffices to go through the first-order correspondents of B2D. I include the proof of (X_{\rightarrow}) because it is not straightforward. Suppose $wR_{\dagger}v$ and $vR_{\circledast}u$. Since R_{\blacksquare} is reflexive, $w \in R_{\blacksquare}[\{w\}]$. Hence, $v \in R_{\dagger}[R_{\blacksquare}[\{w\}]]$, which means $v \in \text{Img}(R_{\circledast})$ by Condition 3-(i). So for some $z \in W$, $zR_{\circledast}v$. By Condition 2-(i), $zR_{\blacksquare}v$. So by Condition 2-(ii), $zR_{\circledast}u$ iff $vR_{\circledast}u$; but by supposition, we have $vR_{\circledast}u$, so $zR_{\circledast}u$. Since R_{\circledast} is a function and $zR_{\circledast}v$, we have $u = v$. □

Lemma 3. *Every point-generated subframe $\mathcal{F}_w \in \text{Fr}_{\mathbf{B2D}}$ is a B-Restall frame.*

Proof. Where $\mathcal{F}_w = \langle W', R'_{\circledast}, R'_{\dagger}, R'_{\blacksquare} \rangle$ is a w -generated subframe in $\text{Fr}_{\mathbf{B2D}}$, we want to establish that \mathcal{F}_w satisfies Conditions 1-3 of B-Restall frames. Condition 1 is immediate from (T_{\blacksquare}) and (5_{\blacksquare}) . Condition 2 follows from three axioms: (1) (F_{\circledast}) , which guarantees that R'_{\circledast} is functional on $\{w \in W'\}$; (2) (G) , which guarantees that R'_{\circledast} is always a subrelation of R'_{\blacksquare} ; and (3) $(\circledast 5_{\blacksquare})$, which guarantees that $R'_{\blacksquare} \circ R'_{\circledast} \subseteq R'_{\circledast}$.¹⁰

Finally, we want to establish that \mathcal{F}_w satisfies Conditions 3-(i) and 3-(ii) of B-Restall frames. For Condition 3-(ii): this follows from Fact 9 (Y_{\leftarrow}) . We begin with Condition 3-(i): $R'_{\dagger}[R'_{\blacksquare}[\{v\}]] = \text{Img}(R'_{\circledast})$. The \subseteq direction follows from (X_{\leftarrow}) . For the \supseteq direction, let v be an arbitrarily chosen world in W' and let d be an arbitrary world in $\text{Img}(R'_{\circledast})$. Let $\hat{w} = f_{\circledast}(w)$.

¹⁰Conditions 1-2 on B-Restall frames are identical to Fritz's (*op. cit.*, pg 392), and the argument identical as well.

We show: (a) $vR'_\blacksquare \circ R'_\dagger \hat{w}$ and (b) $\hat{w}R'_\blacksquare \circ R'_\dagger d$. It will follow by the transitivity of $R'_\blacksquare \circ R'_\dagger$ that $vR'_\blacksquare \circ R'_\dagger d$ and, since d is arbitrary, that $R'_\dagger[R'_\blacksquare[\{v\}]] \supseteq \text{Img}(R'_\circledast)$.

For (a), we note again that by Lemma 1, $\exists u, \hat{v}$ s.t. $wR'_\blacksquare u$ and $uR'_\dagger \hat{v}$ and $\hat{v}R'_\blacksquare v$. Since (by Condition 2) $wR'_\blacksquare \hat{w}$ and R'_\blacksquare is an equivalence relation, it follows that $\hat{w}R'_\blacksquare u$ and $uR'_\dagger \hat{v}$ and $\hat{v}R'_\blacksquare v$. By Fact 10, both \hat{v} and \hat{w} are in D , so by Fact 12 (square completion), $\exists u'$ s.t. $\hat{v}R'_\blacksquare u'R'_\dagger \hat{w}$. But since R'_\blacksquare is an equivalence relation and $\hat{v}R'_\blacksquare u'$, it follows that $vR'_\blacksquare u'$. Hence $vR'_\blacksquare u'R'_\dagger \hat{w}$.

For (b), we note that by Lemma 1, $\exists u, u'$ s.t. $wR'_\blacksquare u$ and $uR'_\dagger u'$ and $u'R'_\blacksquare d$. Since $wR'_\blacksquare \hat{w}$ by Condition 2, it follows that $\hat{w}R'_\blacksquare uR'_\dagger u'R'_\blacksquare d$. By Fact 1, since $u' \in \text{Img}(R'_\dagger)$, $u'R'_\circledast u'$; since $u'R'_\blacksquare d$ and $d \in \text{Img}(R'_\circledast)$, it follows that $u' = d$ by Condition 2. We conclude that $\exists u$: $\hat{w}R'_\blacksquare uR'_\dagger d$.

Moving on to Condition 3-(ii): R_\dagger is reflexive over $\text{Img}(R_\circledast)$. This follows from Fact 9 (Y_\leftarrow).

□

The next step is use $\mathcal{R} \in \mathbf{R}$ to construct a matrix frame \mathcal{M}' which will support a bounded morphism from \mathcal{M}' onto \mathcal{R} . This construction follows the spirit of Fritz (2014), Lemma 2.8: each row of the matrix is constructed from one cell of the R_\blacksquare relation in \mathcal{R} . (I adopt the notational convention that $[x]_E$ is the equivalence class of x under the equivalence relation E .)

Fritz's construction, for \mathcal{R} and \mathcal{M}' with only the R_\blacksquare and R_\circledast -relations, involves proving for each $w \in W_\mathcal{R}$ the existence of a surjection $\alpha_w : W \rightarrow [w]_{R_\blacksquare}$, from $v \in W_\mathcal{R}$ to somewhere in w 's local R_\blacksquare -cell, such that $\alpha_w(w) = f_\circledast(w)$. Intuitively, the function $\alpha_w(\cdot)$ constructs a matrix model \mathcal{M}' row by row. As Fritz says, such an onto function exists for cardinality reasons and because (by $\circledast 5_\blacklozenge$ and (F_\circledast)) there is a unique R_\circledast -fixed point in each R_\blacksquare -cell. The homomorphism h from $W_\mathcal{M}$ to $W_\mathcal{R}$ is given by $h(\langle w, v \rangle) = \alpha_w(v)$ (see Figure 4.)

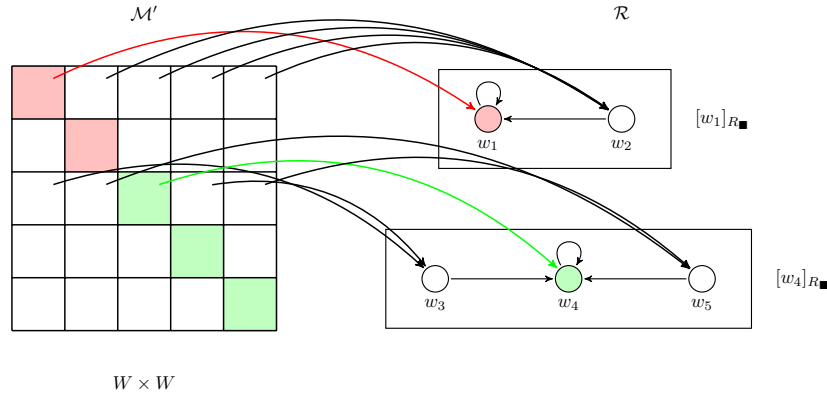


Figure 4: An example of the homomorphism in Fritz (2014). Arrows in the B-Restall frame \mathcal{R} show the R_\circledast relation.

Our version will be similar, but will require a larger matrix for coordinating the R_{\dagger} -relations as well the R_{\blacksquare} - and R_{\circledast} -relations. Lemmas 4-6 and Corollary 1 establish a matrix-like structure within $\mathcal{R} \in \mathbf{R}$. For this purpose, it will be useful to introduce the following notation: where \mathcal{R} is a B-Restall frame, let $C = \{c_d : d \in D\}$ be the set of R_{\blacksquare} -cells in \mathcal{R} . I will use $i, j \dots$ as indices over D and $y, x \dots$ as indices over W .

Lemma 4. $\bigcup_{i \in D} c_i = W$.

Proof. Immediate from the fact that R_{\blacksquare} is an equivalence relation. \square

Lemma 5. *There is a unique R_{\circledast} -fixed point in each R_{\blacksquare} -cell $c_i \subseteq W$.*

Proof. Immediate from Condition 2 on B-Restall frames. Notation: call this R_{\circledast} -fixed point $f_{\circledast}(c_i)$. \square

Lemma 6. *For any c_i and any $c_j \subseteq W$, $\exists w \in c_i, \exists! v \in c_j$ s.t. $wR_{\dagger}v$.*

Proof. Let d_i, d_j be the R_{\circledast} -fixed points of c_i and c_j . By Condition 3 on B-Restall frames and the fact that $d_j \in \text{Img}(R_{\circledast}), \exists w \in W$ s.t. $d_i R_{\blacksquare} w$ and $wR_{\dagger}d_j$. By the definition of $c_i, w \in c_i$. By Lemma 5, d_j is unique. \square

More notation: where $c_i \in C$, let c_i^j be the set $\{w \in c_i : \exists! v \in c_j \text{ such that } wR_{\dagger}v\}$. Let $f_{\dagger}(c_i^j)$ be the unique $v \in c_j$ s.t. $\forall w \in c_i^j, wR_{\dagger}v$. That $f_{\dagger}(c_i^j)$ is a fixed point of R_{\dagger} follows from Fact 5.

Corollary 1. Each c_i can be partitioned into $\{c_i^j : j \in D\}$, where each c_i^j is a nonempty subset of c_i s.t. $\forall w \in c_i^j, wR_{\dagger}(f_{\circledast}(c_j))$. Hence $\bigcup_{j \in D} c_i^j = c_i$, and $\bigcup_{i \in D} (\bigcup_{j \in D} c_i^j) = W$.

Theorem 3. $\mathbf{R} \iff \mathbf{M}$.

Proof. For the right-left direction, we observe that $\mathbf{M} \subseteq \mathbf{R}$.

For the left-right direction: given a B-Restall frame \mathcal{R} with domain W and $\text{Img}(R_{\circledast}) = D$, we will build a matrix frame \mathcal{M} and construct, row-by-row, a surjective bounded morphism g from \mathcal{M} to \mathcal{R} , from which modal equivalence follows (Blackburn et al., 2002, Proposition 2.14).

Let \mathcal{M} be the matrix frame whose set of points is $(D \times W) \times (D \times W)$. For any $i, j \in D$ and $y \in W$, let $A_{i,y}^j$ be the set $\{\langle i, y \rangle\} \times \{\langle j, x \rangle : x \in W\}$. Note that $|A_{i,y}^j| = |W|$.

For each $i, j \in D$ and $y \in W$, fix some $g_{i,y}^j : W \mapsto c_i^j$, an onto function s.t. if $i = j$ then $g_{i,y}^j(y) = f_{\circledast}(c_i)$.

This condition maps the ‘‘diagonal’’ points of the matrix frame into D in the B-Restall frame.

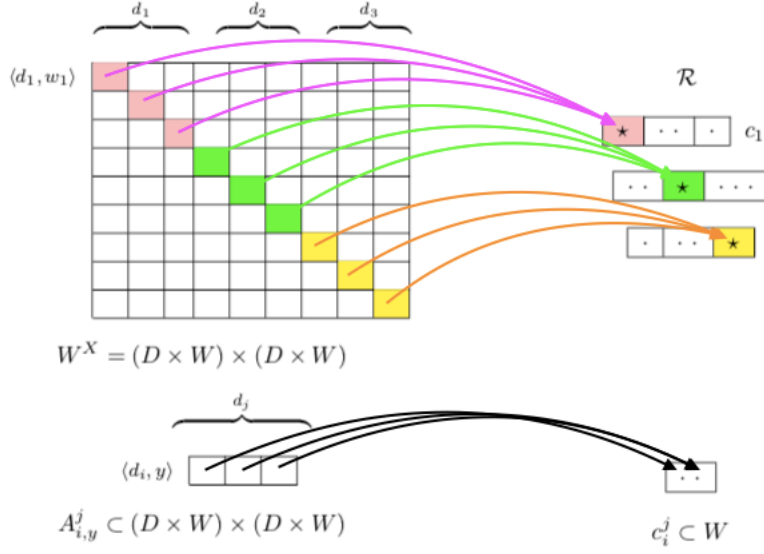


Figure 5: Building a homomorphism F from \mathcal{M} to \mathcal{R} . Stars show the $R_{@}$ -fixed points of \mathcal{R} .

Such onto functions exist for cardinality reasons: $|A^j_{i,y}| \geq |c^j_i|$ because $W \supseteq c^j_i$.

Define $F : (D \times W) \times (D \times W) \mapsto W$ by $F(\langle i, y \rangle, \langle j, w \rangle) = g^j_{i,y}(w)$.

To show that F is a bounded morphism from \mathcal{M} to \mathcal{R} , it suffices to go through the Forth and Back Conditions for $\nabla \in \{\blacksquare, @, \dagger\}$ for each point $\langle \langle i, y \rangle, \langle j, x \rangle \rangle \in (D \times W) \times (D \times W)$, making repeated use of the facts that (i) $g^j_{i,y}(x) \in c^j_i$, (ii) $\forall w \in c^j_i, f_{@}(w) \in c_i$ and $f_{\dagger}(w) \in c_j$, and (iii) $\forall w \in c^j_i$, if $i = j$, then $f_{\dagger}(c^j_i) = f_{@}(c_i)$.

Below, I show the Forth and Back conditions for $\nabla = \blacksquare$, $\nabla = \dagger$, and $\nabla = @$.¹¹

■, Forth: we want to show that if $\langle \langle i, y \rangle, \langle j, x \rangle \rangle R_{\blacksquare}^X \langle \langle i, y \rangle, \langle j', x' \rangle \rangle$, then $F(\langle \langle i, y \rangle, \langle j, x \rangle \rangle) R_{\blacksquare} F(\langle \langle i, y \rangle, \langle j', x' \rangle \rangle)$. This is immediate from the fact that both $F(\langle \langle i, y \rangle, \langle j, x \rangle \rangle)$ and $F(\langle \langle i, y \rangle, \langle j', x' \rangle \rangle)$ are in c_i ; in any B-Restall frame, all $w \in c_i$ are R_{\blacksquare} -related to one another.

■, Back: we want to show that if $F(\langle \langle i, y \rangle, \langle j, x \rangle \rangle) R_{\blacksquare} v'$, then there is some v such that $F(v) = v'$ and $\langle \langle i, y \rangle, \langle j, x \rangle \rangle R_{\blacksquare}^X v$. By surjectivity, each $v' \in c_i$ is such that there is some v such that: $F(\langle \langle i, y \rangle, \langle j', v \rangle \rangle) = v'$. Because $F(\langle \langle i, y \rangle, \langle j', v \rangle \rangle) \in \{\langle i, y \rangle\} \times (D \times W)$, it is guaranteed for any such point that $\langle \langle i, y \rangle, \langle j, x \rangle \rangle R_{\blacksquare} \langle \langle i, y \rangle, \langle j', v \rangle \rangle$.

¹¹The mapping F from $\mathcal{M} = \langle W^X, R_{@}^X, R_{\dagger}^X, R_{\blacksquare}^X, V^X \rangle$ to $\mathcal{R} = \langle W, R_{@}, R_{\dagger}, R_{\blacksquare}, V \rangle$ is a bounded morphism if it satisfies the following conditions (Blackburn et al., 2002, pg. 59):

1. w and $F(w)$ satisfy the same proposition letters;
2. if $w R_{\nabla}^X v$ then $F(w) R_{\nabla} F(v)$ (the *Forth condition*);
3. if $F(w) R_{\nabla} v'$ then there exists some v s.t. $w R_{\nabla}^X v$ and $F(v) = v'$ (the *Back condition*).

We can ensure that condition 1 is met for any valuation function V on \mathcal{R} by ensuring that $F(w)$ satisfies the same proposition-letters as w for any $w \in W_{\mathcal{R}}$.

†, Forth: we want to show that $\langle\langle i, y \rangle, \langle j, x \rangle\rangle R_{\dagger}^X \langle\langle j, x \rangle, \langle j, x \rangle\rangle$ entails $F(\langle i, y \rangle, \langle j, x \rangle) R_{\dagger} F(\langle j, x \rangle, \langle j, x \rangle)$. By construction, $F(\langle i, y \rangle, \langle j, x \rangle) \in c_i^j$ and $F(\langle j, x \rangle, \langle j, x \rangle) \in c_j^j$. Hence $F(\langle i, y \rangle, \langle j, x \rangle) R_{\dagger} F(\langle j, x \rangle, \langle j, x \rangle)$.

†, Back: we want to show that if $F(\langle i, y \rangle, \langle j, x \rangle) R_{\dagger} v'$, then there is some v such that $F(v) = v'$ and $\langle\langle i, y \rangle, \langle j, x \rangle\rangle R_{\dagger}^X v$. By construction, if $F(\langle i, y \rangle, \langle j, x \rangle) R_{\dagger} v'$, then $v' = f_{\textcircled{a}}(c_j)$. A witness for the existential is $\langle\langle j, x \rangle, \langle j, x \rangle\rangle$.

@, Forth: we want to show that $\langle\langle i, y \rangle, \langle j, x \rangle\rangle R_{\textcircled{a}}^X \langle\langle i, y \rangle, \langle i, y \rangle\rangle$ entails $F(\langle i, y \rangle, \langle j, x \rangle) R_{\textcircled{a}} F(\langle i, y \rangle, \langle i, y \rangle)$. For this, it suffices to show that $g_{i,y}^j(x) R_{\textcircled{a}} g_{i,y}^i(y)$. By construction, both $g_{i,y}^j(x)$ and $g_{i,y}^i(y) \in c_i$, and $g_{i,y}^i(y)$ is the $R_{\textcircled{a}}$ -fixed point of c_i . Hence $F(\langle i, y \rangle, \langle j, x \rangle) R_{\textcircled{a}} F(\langle i, y \rangle, \langle i, y \rangle)$.

@, Back: we want to show that if $F(\langle i, y \rangle, \langle j, x \rangle) R_{\textcircled{a}} v'$, then there is some v such that $F(v) = v'$ and $\langle\langle i, y \rangle, \langle j, x \rangle\rangle R_{\textcircled{a}}^X v$. By construction, a witness for this existential is $\langle\langle i, y \rangle, \langle i, y \rangle\rangle$.

□

Theorem 4. B2D is sound and complete for M.

Proof. From Theorems 1-3.

□

C Looking Ahead

The \blacksquare operator is an S5 modal operator, and B2D is a conservative extension of a simple S5 modal logic. In looking ahead to a two-dimensional treatment of other logics, we should ask how two dimensions might be leveraged to upgrade a class of traditional Kripke models whose original accessibility relation(s) capture features of various “flavors” of possibility and necessity.

For example, there is a sense in which the \blacksquare of B2D models an all-inclusive *global* modality, where every world is accessible from every other. This implied globality is why the clause for $\blacksquare\phi$ in §A required no R -restriction. But the accessibility relation for \blacksquare in B2D is *not*, of course, global on the space of world-pairs: for A to remain rigid in the scope of \blacksquare , as (@5 \blacksquare) mandates, \blacksquare *cannot* shift the world-as-actual parameter. Hence whether the two-dimensional entry (\blacksquare) expresses a truly *global* modality, or not, depends on which space—the space of worlds, or the space of world-pairs—one takes to be fundamental to the model.

An *Åqvist logic* (Åqvist, 1973; Segerberg, 1973, pg. 77) can be seen as taking the space of worlds to be fundamental, in the following sense. It interprets the *atoms* of \mathcal{L} as functions from worlds to truth-values that are constant in the world-as-actual parameter—that is, as *propositions* in Stalnaker’s sense.

The class *At* of atomics thus exhibits special features that arbitrary wffs in the full language do not have. Atoms satisfy the restricted schema (#15):

Definition 6 (Segerberg’s Axiom (#15)).

(#15) $\alpha \equiv \dagger\alpha$, for $\alpha \in At$.

Call the constraint imposed by (#15) *barcodeness*: if $f_{\dagger}(w) = f_{\dagger}(w')$, then for atomic α : $w \vDash \alpha$ iff $w' \vDash \alpha$. Intuitively, this corresponds to atomic matrices which are their own diagonalizations.

This commitment, at the level of atoms, to the fundamentality of worlds and propositions is attractive in considering the question in this section: the question of how to upgrade a class of non-S5, one-dimensional Kripke models into stereo. On the Åqvist view, a 1D frame is “lifted” into two dimensions—that is, into the space of matrices—in a way that conservatively extends its profile throughout the one-dimensional fragment of the original language, which is then enriched by A and \dagger . A way to do this is to duplicate any one-dimensional R_∇ relation in the $W \times W$ plane, preserving its frame conditions across the horizontal, and to incorporate axiom schema (#15) into the relevant proof system, thus disallowing non-barcode atoms. The confinement of R_∇ to the horizontal respects (@5 \blacklozenge), the inability of box-diamond modalities to shift the world-as-actual parameter. (#15) limits the expressive power of the propositional fragment to the expressive power of an arbitrary valuation function paired with the original, pre-lifted frame. (#15)’s limitation on closure under substitution thus follows the spirit of e.g. Williamson (2013), who remarks that “the obvious rationale for insisting on ...closure...under uniform substitution in a propositional system is a reading of non-logical sentence letters as *propositional* variables.”¹²

Call such an enriched model a “2D lift” of a one-dimensional Kripke frame, and call the lifted frame enriched with @ and \dagger relations a “Matrix lift” of the same frame. Here, we demonstrate that adding (#15) to our B2D completeness result fits the Åqvist mold for simple S5 frames.

Definition 7. Let B2D + (#15) be the proof system axiomatized by Modus Ponens, Necessitation for each primitive modal operator, Axiom (#15), and the axiom schema

$$(+\text{B2D}) \phi, \text{ where } \phi \text{ is a theorem of B2D.}$$

Note that B2D + (#15) lacks the rule of Uniform Substitution.¹³

Definition 8 (F^\uparrow , the matrix lift of a frame class F). Where F is a class of Kripke frames $\langle W, \{R_{\nabla_i} : i \in I\} \rangle$, the matrix lift of F , F^\uparrow , is the class of matrix models $M^X = \langle W^X, \{R_{\nabla_i} : i \in I\}, V^X \rangle$ s.t.

- $W^X = W \times W$
- $R_{\nabla_i}^X = \{ \langle \langle y, w \rangle, \langle y, w' \rangle \rangle \in W^X \times W^X : w R_{\nabla_i} w' \}$
- $R_{\textcircled{A}}^X = \{ \langle \langle y, w \rangle, \langle y', w' \rangle \rangle \in W^X \times W^X : y = y' \text{ and } w = w' \}$
- $R_{\dagger}^X = \{ \langle \langle y, w \rangle, \langle y', w' \rangle \rangle \in W^X \times W^X : w = w' \text{ and } y = y' \}$
- $V^X : At \mapsto \wp(W^X)$ is such that $\langle y, w \rangle \in V^X(p)$ iff $\langle y', w \rangle \in V^X(p)$

Notation: call such a valuation function V^X a “vertically constant valuation function”.

Theorem 5. B2D + (#15) is sound and complete for $S5^\uparrow$, the matrix lift of the class of S5 Kripke frames.

Proof. Soundness: For (+B2D), since B2D is valid over M and $S5^\uparrow \subset M$, every theorem of B2D is valid over $S5^\uparrow$. For (#15), suppose for some atom a that $a \equiv \dagger a$ is false at some

¹²Williamson (2013, pg. 76, emphasis added); see also Burgess (1999, pg. 176).

¹³For discussion, see, *inter alia*, Smiley (1982); Holliday et al. (2013).

$\langle y, x \rangle \in W_{\mathcal{M}}$, $\mathcal{M} \in \mathbf{S5}^\uparrow$. Then either $\mathcal{M}, y, x \vDash a$ but $\mathcal{M}, x, x \not\vDash a$, or $\mathcal{M}, y, x \not\vDash a$ but $\mathcal{M}, x, x \vDash a$. Either way, \mathcal{M} is not a matrix lifted Kripke frame.

As for the rules: the proof that Modus Ponens preserves validity is straightforward. For Necessitation, suppose that ϕ is valid over $\mathbf{S5}^\uparrow$. Let $\mathcal{M} \in \mathbf{S5}^\uparrow$ and let $y, x \in W$. Then since ϕ is valid over $\mathbf{S5}^\uparrow$, that means for any $x' \in W$: $\mathcal{M}, y, x' \vDash \phi$, so (i) $\mathcal{M}, y, x \vDash \blacksquare\phi$; (ii) $\mathcal{M}, y, y \vDash \phi$, so $\mathcal{M}, y, x \vDash @\phi$; and (iii) $\mathcal{M}, x, x \vDash \phi$, so $\mathcal{M}, y, x \vDash \dagger\phi$. Since y, x were arbitrary, it follows that $\blacksquare\phi$, $@\phi$, and $\dagger\phi$ are all valid over $\mathbf{S5}^\uparrow$. So all the rules preserve validity.

Completeness: suppose Γ is a $\mathbf{B2D} + (\#15)$ -consistent set of formulas. Take $\Delta \supseteq \Gamma$ such that Δ is maximally $\mathbf{B2D} + (\#15)$ -consistent. By Theorem 4, there is a matrix model \mathcal{M} and some $y, x \in W$ such that $\mathcal{M}, y, x \vDash \Delta$. By Necessitation for \blacksquare and \dagger , $\blacksquare\dagger\blacksquare(\alpha \equiv \dagger\alpha) \in \Delta$. Hence, $\mathcal{M}, y, x \vDash \blacksquare\dagger\blacksquare(\alpha \equiv \dagger\alpha)$, which means for any $z, w \in W$: $\mathcal{M}, z, w \vDash \alpha \equiv \dagger\alpha$. So for any $z, z', w \in W$:

$$\begin{aligned}
\langle z, w \rangle \in V^X(p) & \text{ iff } \mathcal{M}z, w \vDash p \\
& \text{ iff } \mathcal{M}, z, w \vDash \dagger p \\
& \text{ iff } \mathcal{M}, w, w \vDash p \\
& \text{ iff } \mathcal{M}, z', w \vDash \dagger p \\
& \text{ iff } \mathcal{M}, z', w \vDash p \\
& \text{ iff } \langle z', w \rangle \in V^X(p).
\end{aligned}$$

Hence, $\mathcal{M} \in \mathbf{S5}^\uparrow$.

□

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