

A Two-Dimensional Logic for Diagonalization and the A Priori*

Melissa Fusco

February 6, 2020

A Introduction

Two-dimensional semantics, which can represent a distinction between apriority and necessity, has wielded considerable influence in the philosophy of language. In this paper, I axiomatize the logic of the † operator of Stalnaker (1978) in the context of two-dimensional modal logic. This operator traces its ancestry through Segerberg (1973) and Åqvist (1973).

In two-dimensional modal logic, the truth of a formula ϕ in a model $\mathcal{M}$ is relativized to two parameters: a parameter representing the *world-as-actual*, which in our semantic notation (as in (@), (†), and (■), below) appears first, and a parameter representing the *world of evaluation*, which we write second.[1] My exposition leans on Stalnaker's notion of *diagonalization*, and his characterization of a priori/necessary ($A/\blacksquare$) distinction as the contrast between the truth of $\blacksquare \dagger \phi$ and the truth of $\blacksquare\phi$. The Kripke models used for this result are shown to be equivalent to Stalnaker's matrix models, which are well known in philosophy of language.

I call this proof system B2D, for "basic two-dimensionalism". The object language contains both "actually" (@) and the † operator.

(@) $\mathcal{M}, y, x \vDash @\phi$ iff $\mathcal{M}, y, y \vDash \phi$

(†) $\mathcal{M}, y, x \vDash \dagger\phi$ iff $\mathcal{M}, x, x \vDash \phi$

The language also contains a global S5 modality ■ common in discussions of two-dimensional semantics (Crossley & Humberstone, 1977; Davies & Humberstone, 1980; Fritz, 2013, 2014):

(■) $\mathcal{M}, y, x \vDash \blacksquare\phi$ iff for all x': $\mathcal{M}, y, x' \vDash \phi$

*This paper benefited from the input of many people. Warm thanks to Peter Fritz, Shawn Standefer, Thomas Icard, Lloyd Humberstone, Jarek Macnar, Michael Nielsen, Wes Holliday, Yifeng Ding, Sridhar Ramesh, Larry Moss, James Walsh, and especially Arc Kocurek.

[1] There is considerable variation in terminology for the first parameter, and indeed whether it should be thought of as a world, or something else (usually richer). Prominently, this parameter corresponds to the *world of the context* in Kaplan (1989), and to the *scenario* parameter in Chalmers (2004).

The focus on †, and the gloss on apriority as ■†, marks a contrast with the classic literature on two-dimensional modal logic, which is centered on @.[23]

Despite this traditional focus, the dagger maintains its unique interest. It is useful to have separate representations of † and ■, for example, to model the distinctive maneuvers of "diagonalizers" (Stalnaker, 1978) and "supervaluers" (Thomason, 1970) in cases where the indeterminacy of the open future affects both the content and the truth-conditions of future-directed discourse (MacFarlane, 2009; Fusco, 2019). In a quite different context, the notion of diagonalization—the interpretation of ϕ as $\dagger\phi$—been used in formal epistemology to model subtleties of Bayesian learning. Stalnaker (2010)'s discussion of the Sleeping Beauty problem (Elga, 2000), for example, suggests that there is a proposition Beauty learns when she wakes up after the coin-toss experiment which is distinct from a superficially indistinguishable proposition she knew before—the key difference being that the first proposition is the diagonalization of the second.[4] These applications are conceptually at home in a framework which treats † on a par with A. Other †-centric work in the literature includes a brief discussion of † in Ch. 10 of Sider (2010), under the name '×', and the discussion in Cresswell (2012), under the name '*Ref*'. Closer to the present study, a completeness theorem for tableau systems featuring @, A, and † (under the name '⊗') appears in Lampert (2018).

In the final section of the paper, I discuss the relationship between the proof system presented here and *Åqvist logics* (Åqvist, 1973; Segerberg, 1973, pg. 77), which intuitively privilege the space of *worlds* over the space of *world-pairs*—and hence the space of *propositions* over the space of general *dipropositions* studied here.[5]

A.1 Two Dimensionalism via †: an outline

Stalnaker's classic discussion in "Assertion" introduces two-dimensional semantics by injecting uncertainty into a situation where speakers express thoughts using context-sensitive language.

> Let me give a simple example: I said *you are a fool* to O'Leary. O'Leary is a fool, so what I said was true, although O'Leary does not think so. Now Daniels, who is no fool and who knows it, was standing near by, and he thought I was talking to him. So both O'Leary and Daniels thought I said something false: O'Leary understood what I said, but disagrees with me about the facts; Daniels, on the other hand, agrees with me about the fact …but misunderstood what I said. Just to fill out the example, let me add that O'Leary believes falsely that Daniels is a fool. Now compare the possible worlds i, j, and k. i is the world as it is …j is the world that O'Leary thinks we are in; and k is the world Daniels thinks we are in…the following TWO-DIMENSIONAL matrix also represents the [first and] second way[s] that the truth-value of my utterance is a function of the facts.

[2]For other work centered around @, see, *inter alia*, Crossley & Humberstone (1977); Gregory (2001); Blackburn & Marx (2002); Hazen et al. (2013); Fritz (2013, 2014).

[3]It is worth noting, in this respect, that apriority (A) can also be expressed via the compound "fixedly actually" ($\mathcal{F}$@) operator discussed by Davies & Humberstone (1980). The semantic entry for $\mathcal{F}$ is:

($\mathcal{F}$) $\mathcal{M}, y, x \vDash \mathcal{F}\phi$ iff for all y': $\mathcal{M}, y', x \vDash \phi$

[4]See the exposition in Magidor (2010). Similar arguments play a role in *screening-off* arguments in decision theory (Ahmed, 2014, Ch. 4.1, Fusco, 2018).

[5]The term *diproposition* comes from Humberstone (1981), whereas *propositional concept* is used by Stalnaker *op. cit.*.

	i	j	k
i	T	F	T
j	T	F	T
k	F	T	F

$p = \ulcorner$*You are a fool*$\urcorner$

$i, i \vDash p$ $\quad$ $j, i \vDash p$ $\quad$ $k, i \nvDash p$

$i, j \nvDash p$ $\quad$ $j, j \nvDash p$ $\quad$ $k, j \vDash p$

$i, k \vDash p$ $\quad$ $j, k \vDash p$ $\quad$ $k, k \nvDash p$

Figure 1: A two-dimensional matrix (figure from Stalnaker (1978)).

In Stalnaker's 3×3 diagram, the vertical axis represents different values of the world-as-actual parameter, while the horizontal axis represents different values of the world of evaluation parameter. In this particular example, the vertical axis disambiguates different resolutions of the context-sensitive expression "you", and the horizontal axis shows worlds i, j, and k in their role as arguments to the relevant propositions expressed. Stalnaker points out that the diagram represents the speaker and O'Leary as agreeing on what *proposition* was expressed by the utterance (since the same row of truth-values, T, F, T, appears to the right of both i and j), even though O'Leary believes what was said was false.[6] That is, the proposition O'Leary thinks was expressed ($i \to$ T, $j \to$ F, $k \to$ T) is false in the world he thinks he is in (viz., j). The fact that the same *column* appears under i and k represents that the speaker and *Daniels* agree on who is a fool and who isn't: both the-world-according-to-the-speaker (viz., i) and the-world-according-to-Daniels (viz., k) agree, for each context-insensitive disambiguation of the "you" in *you are a fool* (respectively: O'Leary in worlds i and j, and Daniels in world k), on whether the proposition expressed by that disambiguated claim is true or false.

The "you are a fool" example involves a misunderstanding of an utterance (on Daniels's part). But sometimes a failure to know how context sensitivity is to be resolved is recognized by all parties. When several candidate worlds are thus ineliminable, a sentence ϕ can be rationally (re)interpreted as $\dagger\phi$: as Stalnaker puts it, the † takes the diagonal of a matrix and "projects" it into the horizontal (*op. cit.*, pg. 82). This is *diagonalization*.

	i	j	k
i	T	F	T
j	T	F	T
k	F	T	F

2D matrix for $\ulcorner$*you are a fool*$\urcorner$

	i	j	k
i	T	F	F
j	T	F	F
k	T	F	F

2D matrix for $\ulcorner\dagger$ (*you are a fool*)$\urcorner$

Figure 2: Diagonalization.

A diagonalized matrix satisfies the condition that the same proposition—the same "horizontal" function from worlds to truth-values—is expressed relative to every candidate for actuality (*op. cit.*, pg. 88).[7] This is a precondition of identifying the *context change potential*, or update of the conversational common ground, associated with the information intuitively carried by an assertion of ϕ (Heim, 1982; Stalnaker, 2002). The terminology of

[6] As is common in the literature, I will move freely between characterizing propositions as sets of possible worlds and as characteristic functions of those sets.

[7] Hawthorne & Magidor (2009) call this feature "Uniformity", noting that diagonalization does not restore Uniformity in non-S5 frameworks (p. 382). Stalnaker himself relates the principle to *Tractatus* Proposition 2.0211 (Wittgenstein, 1974).

the a priori and the a posteriori applies because $\blacksquare\dagger\phi$ represents update on the proposition expressed by $\dagger\phi$, which—unlike the proposition expressed by ϕ—is something interlocutors are in a position to know prior to working out exactly which world they are in. Stalnaker goes on to apply this treatment to the contingent a priori sentence "This bar [the meter bar] is one meter long" (*op. cit.*, pp. 83-84; see Kripke, 1980; Wittgenstein, 1953); the proposition expressed by the sentence might have been false, but since the sentence cannot be falsely uttered, it does not rule out any worlds as candidates for actuality.

Here, we investigate the logic of the two-dimensional language of Crossley & Humberstone (1977), augmented by †. A completeness result, with respect to a class of matrix models for the proof system introduced in the next section, follows closely the strategy of Fritz (2014), §2.

B The logic B2D

Definition 1. Language:
$\mathcal{L} := At \mid (\phi \wedge \phi) \mid \neg\phi \mid \dagger\phi \mid @\phi \mid \blacksquare\phi$
$\blacklozenge$, $\vee$, $\rightarrow$ and $\equiv$ defined as usual; $A\phi$ (*it is a priori that* ϕ) defined as $\blacksquare\dagger\phi$.

Definition 2. Let B2D be the proof system axiomatized by Modus Ponens and Uniform Substitution; all Boolean tautologies; the K Axiom and Necessitation for each primitive modal operator; and the following axioms:

$(T_\blacksquare)$	$\blacksquare p \rightarrow p$	$(F_\dagger)$	$\dagger p \equiv \neg\dagger\neg p$
$(5_\blacksquare)$	$\blacklozenge p \rightarrow \blacksquare\blacklozenge p$	$(X_\rightarrow)$	$\dagger(p \rightarrow @p)$
$(@5_\blacklozenge)$	$\blacklozenge @p \rightarrow @p$	$(Y_\rightarrow)$	$@(p \rightarrow \dagger p)$
(G)	$\blacksquare p \rightarrow @p$	(4_A)	$\blacksquare\dagger p \rightarrow \blacksquare\dagger\blacksquare\dagger p$
$(F_@)$	$@p \equiv \neg @\neg p$	(5_A)	$\blacklozenge\dagger p \rightarrow \blacksquare\dagger\blacklozenge\dagger p$

Notation: $\vdash_{B2D} \phi$ holds if ϕ belongs to the smallest set of formulas that contains all the axioms and is closed under the rules of inference.

While $\mathcal{L}$ augments the language of Crossley & Humberstone (1977) and Fritz (2013, 2014), it is a fragment of the larger language of Segerberg (1973). First we prove some Facts in B2D that are familiar from this literature.

Fact 1. $\vdash_{B2D} \dagger(p \equiv @p)$

Proof. The right-left direction of the embedded biconditional is an axiom.

We reason with the following equivalences:

1.	$\dagger(\neg p \rightarrow @\neg p)$	instance of $(X_\rightarrow)$
2.	$\dagger(\neg @\neg p \rightarrow p)$	from (1), Contraposition
3.	$\dagger(@p \rightarrow p)$	from (2), $(F_@)$, DNE
4.	$\dagger(@p \equiv p)$	from (3), $(X_\rightarrow)$, Necessitation and K Axiom for †

Call this theorem (X) and call the theorem $\dagger(@p \rightarrow p)$, which was proved at Step 3 above, $(X_\leftarrow)$. □

Fact 2. $\vdash_{B2D} @(p \equiv \dagger p)$

Proof. The right-left direction is an axiom. The left-right direction is immediate from a proof similar to the proof of Fact 1. Call this theorem (Y), and call the theorem $@(\dagger p \to p)$, which was proved at Step 3 in the analogous proof to the proof of Fact 1 above, $(Y_{\leftarrow})$. $\square$

Fact 3. $\vdash_{B2D} @p \to \blacksquare @p$

Proof. From $(@5_{\blacklozenge})$ and $(F_{@})$. Call this theorem $(@5_{\blacksquare})$ (Crossley & Humberstone, 1977, pg. 14).[8] $\square$

Fact 4. $\vdash_{B2D} \dagger\blacklozenge\dagger p \equiv \blacklozenge\dagger p$

Proof. The left-right direction is immediate from (4_A) and $(T_{\blacksquare})$. The right-left direction is immediate from (5_A) and $(T_{\blacksquare})$. Call this biconditional $(\dagger\blacklozenge\dagger)$, or the *columnarity* of $R_{\blacksquare}$. $\square$

Fact 5. $\vdash_{B2D} \dagger\dagger p \equiv \dagger p$

Proof. We reason with the following equivalences:

1.	$\dagger @p \equiv \dagger p$	from Fact 1, K axiom for $\dagger$
2.	$@\dagger p \equiv @p$	from Fact 2, K axiom for $@$
3.	$\dagger @\dagger p \equiv \dagger\dagger p$	from (1) (substituting p with $\dagger p$)
4.	$\dagger @\dagger p \equiv \dagger @p$	from (2), Necessitation with $\dagger$
5.	$\dagger\dagger p \equiv \dagger p$	from (1), (3), (4), Transitivity

Call this $(\mathrm{Red}_{\dagger})$. $\square$

Fact 6. $\vdash_{B2D} @@p \equiv @p$

Proof. This can be proven analogously to the proof of Fact 5. $\square$

Call this $(\mathrm{Red}_{@})$.

Fact 7. $\vdash_{B2D} \blacksquare\dagger p \to @p$

Proof. From (G) and (Y) (viz., Fact 2). By instantiating '$\dagger p$' for p, we get $\blacksquare\dagger p \to @\dagger p$ from (G). By distributing the @ operator over (Y), we get $@\dagger p \equiv @p$. Hence $\blacksquare\dagger p \to @p$ by transitivity of $\to$.

Call this $(A@)$. $\square$

Summarizing these facts:

[8] Because Crossley & Humberstone use 'A' for "actually" and '$\square$' for our $\blacksquare$, their version is called '$(A5_{\square})$'.

(X) $\dagger(p \equiv @p)$

$(X_{\leftarrow})$ $\dagger(@p \to p)$

(Y) $@(p \equiv \dagger p)$

$(Y_{\leftarrow})$ $@(\dagger p \to p)$

$(@5_{\blacksquare})$ $@p \to \blacksquare @p$

$(\blacklozenge \dagger \blacklozenge)$ $\dagger \blacklozenge \dagger p \equiv \blacklozenge \dagger p$

$(\text{Red}_{\dagger})$ $\dagger \dagger p \equiv \dagger p$

$(\text{Red}_{@})$ $@@p \equiv @p$

$(A@)$ $\blacksquare \dagger p \to @p$

B.1 The class $\mathsf{Fr}_{\mathsf{B2D}}$

Definition 3. A **Kripke frame** is a tuple $\mathcal{K} = \langle W, \{R_{\triangledown_i} : i \in I\}\rangle$ where W is a set of points and $R_{\triangledown_i}, i \in I$ are binary relations on W.

Fact 8. The formulas in B2D have the following global first-order correspondents in Kripke frames (where '$uR_{\blacksquare} \circ R_{\dagger}v$' abbreviates relation $\exists z : uR_{\blacksquare}zR_{\dagger}v$):

$(T_{\blacksquare})$ $R_{\blacksquare}$ is reflexive

$(5_{\blacksquare})$ $R_{\blacksquare}$ is Euclidean

$(@5_{\blacklozenge})$ $\forall w, v, u : (wR_{\blacksquare}u \wedge wR_{@}v) \to uR_{@}v$

(G) $R_{@}$ is a subrelation of $R_{\blacksquare}$

$(F_{\dagger})$ $R_{\dagger}$ is a function

$(F_{@})$ $R_{@}$ is a function

$(X_{\to})$ $\forall w, v, u : (wR_{\dagger}v \wedge vR_{@}u \to v = u)$

$(Y_{\to})$ $\forall w, v, u : (wR_{@}v \wedge vR_{\dagger}u \to v = u)$

(4_A) $R_{\blacksquare} \circ R_{\dagger}$ is transitive

(5_A) $\forall x, y, a, b[(xR_{\blacksquare}y \wedge xR_{\blacksquare}a \wedge aR_{\dagger}b) \to \exists c(bR_{\blacksquare}c \wedge \forall d(cR_{\dagger}d \to yR_{\dagger}d))]$

Fact 9. Additionally, theorems $(X_{\leftarrow})$, (X), $(Y_{\leftarrow})$, (Y), and $(@5_{\blacklozenge})$ (c.f. Facts 1-3) have the following local first-order correspondents in Kripke frames:

$(X_{\leftarrow})$ within $Img(R_{\dagger})$, $R_{@}$ is reflexive

$(Y_{\leftarrow})$ within $Img(R_{@})$, $R_{\dagger}$ is reflexive

$(@5_{\blacksquare})$ $R_{\blacksquare} \circ R_{@} \subseteq R_{@}$

(X) within $Img(R_{\dagger})$, $R_{@}$ is the identity relation

(Y) within $Img(R_{@})$, $R_{\dagger}$ is the identity relation

Definition 4. A **basic matrix frame** $\mathcal{M}$ is a Kripke frame $\langle W^X, R^X_{\blacksquare}, R^X_{@}, R^X_{\dagger}\rangle$, where $W^X = X \times X$ for some nonempty set X and:

- $\langle y, x\rangle R^X_{\blacksquare} \langle y', x'\rangle$ iff $y = y'$
- $\langle y, x\rangle R^X_{@} \langle y', x'\rangle$ iff $y' = x' = y$.
- $\langle y, x\rangle R^X_{\dagger} \langle y', x'\rangle$ iff $y' = x' = x$.

The guiding spirit of the completeness proof for the class of basic matrix frames comes from Fritz (2014, §1-2). Following that proof, we identify three types of structure, $\mathsf{M} \subsetneq \mathsf{R} \subsetneq \mathsf{Fr}_{\mathsf{B2D}}$.

1. $\mathsf{Fr}_{\mathsf{B2D}}$, the class of Kripke frames on which B2D is valid;
2. R, an intermediate class of "B-Restall frames"[9];

[9] So-called because of their similarity to the frames in Restall (2012).

3. M (the target class), the class of basic matrix frames.

$\mathsf{Fr}_{\mathsf{B2D}}$ is the class of Kripke frames for which soundness and completeness for the B2D axioms is immediate via Sahlqvist's Completeness Theorem (Blackburn et al., 2002, Ch. 4). The second class, R, is the class of point-generated subframes of $\mathsf{Fr}_{\mathsf{B2D}}$. The smallest class, M, is the class of basic matrix frames, identical to those used by Stalnaker in his discussion of the "you are a fool" and "this bar is one meter long" cases.

We seek to show that these classes are modally equivalent ($\mathsf{Fr}_{\mathsf{B2D}} \leftrightsquigarrow \mathsf{R} \leftrightsquigarrow \mathsf{M}$): that is, that the same sets of formulas are satisfiable in each. To do so, we use two invariance results for modal models: (i) $\mathsf{Fr}_{\mathsf{B2D}} \leftrightsquigarrow \mathsf{R}$ because modal satisfaction is invariant under generated submodels; and (ii) $\mathsf{R} \leftrightsquigarrow \mathsf{M}$ because modal satisfaction is invariant under bounded morphisms.

$\mathsf{Fr}_{\mathsf{B2D}}$ ········ gen. submodel ········ R - - - - bounded morphism - - - - M

Notation: for a relation $R_\triangledown$ that is a function, I will use "$f_\triangledown(x)$" for the unique y s.t. $xR_\triangledown y$. Let $D = Img(R_@)$. I will use d_1, d_2, etc. for arbitrary worlds in D.

Now for some additional frame conditions, imposed by the axiom schemata on $\mathcal{F} \in \mathsf{Fr}_{\mathsf{B2D}}$:

Fact 10 (Alternative characterization of diagonal points)**.** $Img(R_@) = Img(R_\dagger)$.

Proof. Let $u \in Img(R_@)$. Hence by $(Y_\leftarrow)$, $u \in Img(R_\dagger)$. Conversely, let $u \in Img(R_\dagger)$. Hence by $(X_\leftarrow)$, $u \in Img(R_@)$. □

Fact 11. $R_\blacksquare \circ R_\dagger$ is Euclidean.

Proof. For brevity, let $R_A := R_\blacksquare \circ R_\dagger$. Suppose xR_Ay and xR_Az. Thus, there is a y' and z' such that $xR_\blacksquare y'$ and $y'R_\dagger y$, and also $xR_\blacksquare z'$ and $z'R_\dagger z$. Since $xR_\blacksquare y'$ and $xR_\blacksquare z'$ and $z'R_\dagger z$, by (5_A), there is a z'' such that $zR_\blacksquare z''$ and for all u, if $z''R_\dagger u$, then $y'R_\dagger u$. Now, within $Img(R_\dagger)$, $R_\dagger$ is the identity relation (by (X) and Fact 10.) Hence, if $y'R_\dagger u$, then $u = y$. Therefore, if $z''R_\dagger u$, then $u = y$. Since $R_\dagger$ is a total function, that means $z''R_\dagger y$. Hence, we have $zR_\blacksquare z''$ and $z''R_\dagger y$. So, zR_Ay. □

Fact 12 (Square Completion (Symmetry) in $Img(R_@)$)**.** $\forall d_1, d_2 \in Img(R_@)$: if $d_1 R_\blacksquare \circ R_\dagger d_2$, then $d_2 R_\blacksquare \circ R_\dagger d_1$.

Proof. Suppose $d_1 \in Img(R_@)$. Then $\exists z : d_1 R_\blacksquare z R_\dagger d_1$: $R_\blacksquare$ is reflexive and $R_\dagger$ is reflexive within $Img(R_@)$. By assumption, $\exists z : d_1 R_\blacksquare z R_\dagger d_2$. It follows by (5_A), the Euclideanness of $R_\blacksquare \circ R_\dagger$, that $\exists z' d_2 R_\blacksquare z' R_\dagger d_1$. □

Fact 13 ($R_\blacksquare \circ R_\dagger$ an equivalence relation on $Img(R_@)$)**.** $R_\blacksquare \circ R_\dagger$ an equivalence relation on $Img(R_@)$.

Proof. This follows from the fact that within $Img(R_@)$, $R_\blacksquare \circ R_\dagger$ is symmetric (by Fact 12), transitive, and reflexive. □

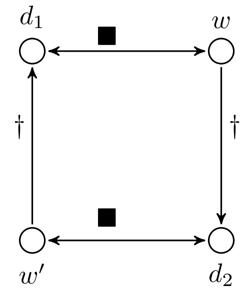

Figure 3: Facts 12 (Square completion) and 14 (Mirror points).

Fact 14 ($A\dagger$ Mirror Points)**.** Suppose $wR_@d_1$ and $wR_\dagger d_2$. Then $\exists w'$ s.t. $w'R_@d_2$ and $w'R_\dagger d_1$.

Proof. Suppose $wR_@d_1$ and $wR_\dagger d_2$. It follows (from $(X_\leftarrow)$) that both $d_1, d_2 \in Img(R_@)$. Because $wR_@d_1$, $wR_\blacksquare d_1$ and so $d_1R_\blacksquare w$ by symmetry of $R_\blacksquare$. Hence $\exists z : d_1R_\blacksquare zR_\dagger d_2$ as witnessed by w. By Fact 12, $\exists z : d_2R_\blacksquare zR_\dagger d_1$. Let w' be the z such that $d_2R_\blacksquare z \wedge zR_\dagger d_1$. Since $w'R_\blacksquare d_2$ and $d_2R_@d_2$, by $(@5_\blacklozenge)$ $w'R_@d_2$. Hence $w'R_@d_2$ and $w'R_\dagger d_1$. □

Theorem 1. B2D *is sound and strongly complete with respect to* $\mathsf{Fr}_{\mathsf{B2D}}$*, the class of* B2D*-frames.*

Proof. By the Sahlqvist Completeness Theorem (Blackburn et al., 2002, Ch. 4).

□

B.2 The class R

We seek to show that this class is modally equivalent to the class of B-Restall frames—that a set of formulas is satisfiable on $\mathsf{Fr}_{\mathsf{B2D}}$ iff it is satisfiable on R, the class of B-Restall Frames. To do so, it will be helpful to prove a Lemma concerning the point-generated universe.

Notation: in general, we write $R[Y]$ for the image of a set Y under R.

Lemma 1. *For any point-generated subframe* $\mathcal{F}'_w = \langle W', R'_\blacksquare, R'_\dagger, R'_@\rangle \in \mathsf{Fr}_{\mathsf{B2D}}$*:* $W' = R_\blacksquare[R_\dagger[R_\blacksquare[\{w\}]]]$.

Proof. We let $X := R_\blacksquare[R_\dagger[R_\blacksquare[\{w\}]]]$. We show (1) $w \in X$ and (2) that for each operator $\triangledown$ that $R_\triangledown[X] \subseteq X$.
(1) Let $u = f_@(w)$. By (G), $wR_\blacksquare u$. By (Y), $uR_\dagger u$. By symmetry of $R_\blacksquare$, $uR_\blacksquare w$. Hence via u, it follows that $w \in X$.
(2-■). Since $R_\blacksquare$ is transitive and reflexive, $R_\blacksquare[X] = X$.
(2-†). Let $v \in R_\dagger[X]$. Now $R_\dagger[X] = R_\dagger[R_\blacksquare[R_\dagger[R_\blacksquare[\{w\}]]]]$, by the definition of X. By (4_A), $R_\dagger[R_\blacksquare[R_\dagger[R_\blacksquare[\{w\}]]]] = R_\dagger[R_\blacksquare[\{w\}]$. Hence it suffices to show that if $v \in R_\dagger[R_\blacksquare[\{w\}]$, then $v \in R_\blacksquare[R_\dagger[R_\blacksquare[\{w\}]]]$. This follows immediately from $(T_\blacksquare)$: viz., from the fact that $vR_\blacksquare v$.
(2-@). Let $v \in R_@[X]$. By Fact 7, $v \in R_\dagger[R_\blacksquare[X]]$ and so $v \in R_\dagger[R_\blacksquare[R_\blacksquare[R_\dagger[R_\blacksquare[\{w\}]]]]]$. By Transitivity of $R_\blacksquare$, $v \in R_\dagger[R_\blacksquare[R_\dagger[R_\blacksquare[\{w\}]]]]$. By (4_A), $v \in R_\dagger[R_\blacksquare[\{w\}]$. From $(T_\blacksquare)$, we again conclude that $v \in R_\blacksquare[R_\dagger[R_\blacksquare[\{w\}]]$. □

Definition 5. A **B-Restall Frame** is a frame $\mathcal{R} = \langle W, R_\blacksquare, R_@, R_\dagger \rangle$ satisfying three conditions:

1. $R_\blacksquare$ is an equivalence relation
2. $R_@$ is a function such that
   (i) $wR_@v \to wR_\blacksquare v$
   (ii) $R_@$ maps any two $R_\blacksquare$-related worlds to the same point
3. $R_\dagger$ is a function such that
   (i) for any w: $R_\dagger[R_\blacksquare[\{w\}]] = Img(R_@)$
   (ii) $R_\dagger$ is reflexive over $Img(R_@)$

Let R be the class of B-Restall frames. Let D be the set $Img(R_@)$ in a B-Restall frame.

Theorem 2. $\mathsf{Fr}_{\mathsf{B2D}} \leftrightsquigarrow \mathsf{R}$.

Theorem 2 is an consequence of the Lemmas 2 and 3, below. By Lemma 2, any formula falsifiable in $\mathcal{R} \in \mathsf{R}$ is falsifiable in some $\mathcal{F} \in \mathsf{Fr}_{\mathsf{B2D}}$. By Lemma 3, any formula falsifiable in a point-generated subframe $\mathcal{F}_w \in \mathsf{Fr}_{\mathsf{B2D}}$ is falsifiable in some $\mathcal{R} \in \mathsf{R}$.

Because modal satisfaction is invariant under point-generated submodels (Blackburn et al., 2002, Proposition 2.6.), the formulas falsifiable in $\mathcal{R} \in \mathsf{R}$ are *exactly* the formulas falsifiable in $\mathcal{F} \in \mathsf{Fr}_{\mathsf{B2D}}$.

Lemma 2. $\mathsf{R} \subseteq \mathsf{Fr}_{\mathsf{B2D}}$.

Proof. To verify this, it suffices to go through the first-order correspondents of B2D. I include the proof of $(X_\rightarrow)$ because it is not straightforward. Suppose $wR_\dagger v$ and $vR_@u$. Since $R_\blacksquare$ is reflexive, $w \in R_\blacksquare[\{w\}]$. Hence, $v \in R_\dagger[R_\blacksquare[\{w\}]]$, which means $v \in Img(R_@)$ by Condition 3-(i). So for some $z \in W$, $zR_@v$. By Condition 2-(i), $zR_\blacksquare v$. So by Condition 2-(ii), $zR_@u$ iff $vR_@u$; but by supposition, we have $vR_@u$, so $zR_@u$. Since $R_@$ is a function and $zR_@v$, we have $u = v$.

□

Lemma 3. *Every point-generated subframe $\mathcal{F}_w \in \mathsf{Fr}_{\mathsf{B2D}}$ is a B-Restall frame.*

Proof. Where $\mathcal{F}_w = \langle W', R'_@, R'_\dagger, R'_\blacksquare \rangle$ is a w-generated subframe in $\mathsf{Fr}_{\mathsf{B2D}}$, we want to establish that $\mathcal{F}_w$ satisfies Conditions 1-3 of B-Restall frames. Condition 1 is immediate from $(T_\blacksquare)$ and $(5_\blacksquare)$. Condition 2 follows from three axioms: (1) $(F_@)$, which guarantees that $R'_@$ is functional on $\{w \in W'\}$; (2) (G), which guarantees that $R'_@$ is always a subrelation of $R'_\blacksquare$; and (3) $(@5_\blacksquare)$, which guarantees that $R'_\blacksquare \circ R'_@ \subseteq R'_@$.[10]

Finally, we want to establish that $\mathcal{F}_w$ satisfies Conditions 3-(i) and 3-(ii) of B-Restall frames. For Condition 3-(ii): this follows from Fact 9 $(Y_\leftarrow)$. We begin with Condition 3-(i): $R'_\dagger[R'_\blacksquare[\{v\}]] = Img(R'_@)$. The $\subseteq$ direction follows from $(X_\leftarrow)$. For the $\supseteq$ direction, let v be an arbitrarily chosen world in W' and let d be an arbitrary world in $Img(R'_@)$. Let $\hat{w} = f_@(w)$.

[10] Conditions 1-2 on B-Restall frames are identical to Fritz's (*op. cit.*, pg 392), and the argument identical as well.

We show: (a) $vR'_{\blacksquare} \circ R'_{\dagger}\hat{w}$ and (b) $\hat{w}R'_{\blacksquare} \circ R'_{\dagger}d$. It will follow by the transitivity of $R'_{\blacksquare} \circ R'_{\dagger}$ that $vR'_{\blacksquare} \circ R'_{\dagger}d$ and, since d is arbitrary, that $R'_{\dagger}[R'_{\blacksquare}[\{v\}]] \supseteq Img(R'_{@})$.

For (a), we note again that by Lemma 1, $\exists u, \hat{v}$ s.t. $wR'_{\blacksquare}u$ and $uR'_{\dagger}\hat{v}$ and $\hat{v}R'_{\blacksquare}v$. Since (by Condition 2) $wR'_{\blacksquare}\hat{w}$ and $R'_{\blacksquare}$ is an equivalence relation, it follows that $\hat{w}R'_{\blacksquare}u$ and $uR'_{\dagger}\hat{v}$ and $\hat{v}R'_{\blacksquare}v$. By Fact 10, both $\hat{v}$ and $\hat{w}$ are in D, so by Fact 12 (square completion), $\exists u'$ s.t. $\hat{v}R'_{\blacksquare}u'R_{\dagger}\hat{w}$. But since $R'_{\blacksquare}$ is an equivalence relation and $\hat{v}R'_{\blacksquare}u'$, it follows that $vR'_{\blacksquare}u'$. Hence $vR'_{\blacksquare}u'R'_{\dagger}\hat{w}$.

For (b), we note that by Lemma 1, $\exists u, u'$ s.t. $wR'_{\blacksquare}u$ and $uR'_{\dagger}u'$ and $u'R'_{\blacksquare}d$. Since $wR'_{\blacksquare}\hat{w}$ by Condition 2, it follows that $\hat{w}R'_{\blacksquare}uR'_{\dagger}u'R'_{\blacksquare}d$. By Fact 1, since $u' \in Img(R'_{\dagger})$, $u'R'_{@}u'$; since $u'R'_{\blacksquare}d$ and $d \in Img(R'_{@})$, it follows that $u' = d$ by Condition 2. We conclude that $\exists u$: $\hat{w}R'_{\blacksquare}uR'_{\dagger}d$.

Moving on to Condition 3-(ii): $R_{\dagger}$ is reflexive over $Img(R_{@})$. This follows from Fact 9 $(Y_{\leftarrow})$.

□

The next step is use $\mathcal{R} \in \mathsf{R}$ to construct a matrix frame $\mathcal{M}'$ which will support a bounded morphism from $\mathcal{M}'$ onto $\mathcal{R}$. This construction follows the spirit of Fritz (2014), Lemma 2.8: each row of the matrix is constructed from one cell of the $R_{\blacksquare}$ relation in $\mathcal{R}$. (I adopt the notational convention that $[x]_E$ is the equivalence class of x under the equivalence relation E.)

Fritz's construction, for $\mathcal{R}$ and $\mathcal{M}'$ with only the $R_{\blacksquare}$ and $R_{@}$-relations, involves proving for each $w \in W_{\mathcal{R}}$ the existence of a surjection $\alpha_w : W \to [w]_{R_{\blacksquare}}$, from $v \in W_{\mathcal{R}}$ to somewhere in w's local $R_{\blacksquare}$-cell, such that $\alpha_w(w) = f_{@}(w)$. Intuitively, the function $\alpha_w(\cdot)$ constructs a matrix model $\mathcal{M}'$ row by row. As Fritz says, such an onto function exists for cardinality reasons and because (by $(@5_{\blacklozenge})$ and $(F_{@})$) there is a unique $R_{@}$-fixed point in each $R_{\blacksquare}$-cell. The homomorphism h from $W_{\mathcal{M}}$ to $W_{\mathcal{R}}$ is given by $h(\langle w, v\rangle) = \alpha_w(v)$ (see Figure 4.)

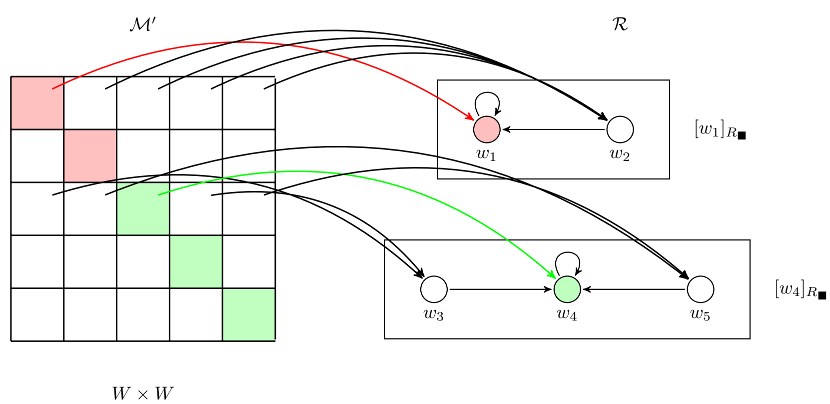

Figure 4: An example of the homomorphism in Fritz (2014). Arrows in the B-Restall frame $\mathcal{R}$ show the $R_{@}$ relation.

Our version will be similar, but will require a larger matrix for coordinating the $R_\dagger$-relations as well the $R_\blacksquare$- and $R_@$-relations. Lemmas 4-6 and Corollary 1 establish a matrix-like structure within $\mathcal{R} \in \mathsf{R}$. For this purpose, it will be useful to introduce the following notation: where $\mathcal{R}$ is a B-Restall frame, let $C = \{c_d : d \in D\}$ be the set of $R_\blacksquare$-cells in $\mathcal{R}$. I will use $i, j \ldots$ as indices over D and $y, x \ldots$ as indices over W.

Lemma 4. $\bigcup_{i \in D} c_i = W$.

Proof. Immediate from the fact that $R_\blacksquare$ is an equivalence relation. □

Lemma 5. *There is a unique $R_@$-fixed point in each $R_\blacksquare$-cell $c_i \subseteq W$.*

Proof. Immediate from Condition 2 on B-Restall frames. Notation: call this $R_@$-fixed point $f_@(c_i)$. □

Lemma 6. *For any c_i and any $c_j \subseteq W$, $\exists w \in c_i, \exists! v \in c_j$ s.t. $wR_\dagger v$.*

Proof. Let d_i, d_j be the $R_@$-fixed points of c_i and c_j. By Condition 3 on B-Restall frames and the fact that $d_j \in Img(R_@), \exists w \in W$ s.t. $d_i R_\blacksquare w$ and $wR_\dagger d_j$. By the definition of $c_i, w \in c_i$. By Lemma 5, d_j is unique. □

More notation: where $c_i \in C$, let c_i^j be the set $\{w \in c_i : \exists! v \in c_j$ such that $wR_\dagger v\}$. Let $f_\dagger(c_i^j)$ be the unique $v \in c_j$ s.t. $\forall w \in c_i^j$, $wR_\dagger v$. That $f_\dagger(c_i^j)$ is a fixed point of $R_\dagger$ follows from Fact 5.

Corollary 1. Each c_i can be partitioned into $\{c_i^j : j \in D\}$, where each c_i^j is a nonempty subset of c_i s.t. $\forall w \in c_i^j$, $wR_\dagger(f_@(c_j))$. Hence $\bigcup_{j \in D} c_i^j = c_i$, and $\bigcup_{i \in D}(\bigcup_{j \in D} c_i^j) = W$.

Theorem 3. $\mathsf{R} \leftrightsquigarrow \mathsf{M}$.

Proof. For the right-left direction, we observe that $\mathsf{M} \subseteq \mathsf{R}$.

For the left-right direction: given a B-Restall frame $\mathcal{R}$ with domain W and $Img(R_@) = D$, we will build a matrix frame $\mathcal{M}$ and construct, row-by-row, a surjective bounded morphism g from $\mathcal{M}$ to $\mathcal{R}$, from which modal equivalence follows (Blackburn et al., 2002, Proposition 2.14).

Let $\mathcal{M}$ be the matrix frame whose set of points is $(D \times W) \times (D \times W)$. For any $i, j \in D$ and $y \in W$, let $A_{i,y}^j$ be the set $\{\langle i, y\rangle\} \times \{\langle j, x\rangle : x \in W\}$. Note that $|A_{i,y}^j| = |W|$.

For each $i, j \in D$ and $y \in W$, fix some $g_{i,y}^j : W \mapsto c_i^j$, an onto function s.t. if $i = j$ then $g_{i,y}^j(y) = f_@(c_i)$.

This condition maps the "diagonal" points of the matrix frame into D in the B-Restall frame.

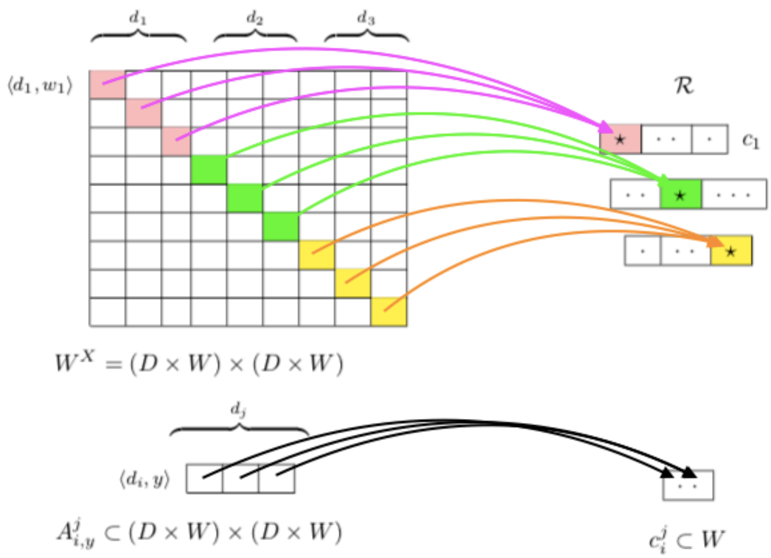

Figure 5: Building a homomorphism F from $\mathcal{M}$ to $\mathcal{R}$. Stars show the $R_@$-fixed points of $\mathcal{R}$.

Such onto functions exist for cardinality reasons: $|A^j_{i,y}| \geq |c^j_i|$ because $W \supseteq c^i_j$.

Define $F : (D \times W) \times (D \times W) \mapsto W$ by $F(\langle i, y\rangle, \langle j, w\rangle) = g^j_{i,y}(w)$.

To show that F is a bounded morphism from $\mathcal{M}$ to $\mathcal{R}$, it suffices to go through the Forth and Back Conditions for $\triangledown \in \{\blacksquare, @, \dagger\}$ for each point $\langle\langle i, y\rangle, \langle j, x\rangle\rangle \in (D \times W) \times (D \times W)$, making repeated use of the facts that (i) $g^j_{i,y}(x) \in c^j_i$, (ii) $\forall w \in c^j_i, f_@(w) \in c_i$ and $f_\dagger(w) \in c_j$, and (iii) $\forall w \in c^i_j$, if $i = j$, then $f_\dagger(c^j_i) = f_@(c_i)$.

Below, I show the Forth and Back conditions for $\triangledown = \blacksquare$, $\triangledown = \dagger$, and $\triangledown = @$.[11]

$\blacksquare$, Forth: we want to show that if $\langle\langle i, y\rangle, \langle j, x\rangle\rangle R^X_\blacksquare \langle\langle i, y\rangle, \langle j', x'\rangle\rangle$, then $F(\langle i, y\rangle, \langle j, x\rangle) R_\blacksquare F(\langle i, y\rangle, \langle j', x'\rangle)$. This is immediate from the fact that both $F(\langle i, y\rangle, \langle j, x\rangle)$ and $F(\langle i, y\rangle, \langle j', x'\rangle)$ are in c_i; in any B-Restall frame, all $w \in c_i$ are $R_\blacksquare$-related to one another.

$\blacksquare$, Back: we want to show that if $F(\langle i, y\rangle, \langle j, x\rangle) R_\blacksquare v'$, then there is some v such that $F(v) = v'$ and $\langle\langle i, y\rangle, \langle j, x\rangle\rangle R^X_\blacksquare v$. By surjectivity, each $v' \in c^j_i$ is such that there is some v such that: $F(\langle\langle i, y\rangle, \langle j', v\rangle\rangle) = v'$. Because $F(\langle\langle i, y\rangle, \langle j', v\rangle\rangle) \in \{\langle i, y\rangle\} \times (D \times W)$, it is guaranteed for any such point that $\langle\langle i, y\rangle, \langle j, x\rangle\rangle R_\blacksquare \langle\langle i, y\rangle, \langle j', v\rangle\rangle$.

[11] The mapping F from $\mathcal{M} = \langle W^X, R^X_@, R^X_\dagger, R^X_\blacksquare, V^X\rangle$ to $\mathcal{R} = \langle W, R_@, R_\dagger, R_\blacksquare, V\rangle$ is a bounded morphism if it satisfies the following conditions (Blackburn et al., 2002, pg. 59):

1. w and $F(w)$ satisfy the same proposition letters;
2. if $w R^X_\triangledown v$ then $F(w) R_\triangledown F(v)$ (the *Forth condition*);
3. if $F(w) R_\triangledown v'$ then there exists some v s.t. $w R^X_\triangledown v$ and $F(v) = v'$ (the *Back condition*).

We can ensure that condition 1 is met for any valuation function V on $\mathcal{R}$ by ensuring that $F(w)$ satisfies the same proposition-letters as w for any $w \in W_\mathcal{R}$.

†, Forth: we want to show that $\langle\langle i,y\rangle,\langle j,x\rangle\rangle R^X_\dagger \langle\langle j,x\rangle,\langle j,x\rangle\rangle$ entails $F(\langle\langle i,y\rangle,\langle j,x\rangle\rangle) R_\dagger F(\langle\langle j,x\rangle,\langle j,x\rangle\rangle)$. By construction, $F(\langle\langle i,y\rangle,\langle j,x\rangle\rangle) \in c^j_i$ and $F(\langle\langle j,x\rangle,\langle j,x\rangle\rangle) \in c^j_j$. Hence $F(\langle\langle i,y\rangle,\langle j,x\rangle\rangle) R_\dagger F(\langle\langle j,x\rangle,\langle j,x\rangle\rangle)$.

†, Back: we want to show that if $F(\langle\langle i,y\rangle,\langle j,x\rangle\rangle) R_\dagger v'$, then there is some v such that $F(v) = v'$ and $\langle\langle i,y\rangle,\langle j,x\rangle\rangle R^X_\dagger v$. By construction, if $F(\langle\langle i,y\rangle,\langle j,x\rangle\rangle) R_\dagger v'$, then $v' = f_@(c_j)$. A witness for the existential is $\langle\langle j,x\rangle,\langle j,x\rangle\rangle$.

@, Forth: we want to show that $\langle\langle i,y\rangle,\langle j,x\rangle\rangle R^X_@ \langle\langle i,y\rangle,\langle i,y\rangle\rangle$ entails $F(\langle\langle i,y\rangle,\langle j,x\rangle\rangle) R_@ F(\langle\langle i,y\rangle,\langle i,y\rangle\rangle)$. For this, it suffices to show that $g^j_{i,y}(x) R_@ g^i_{i,y}(y)$. By construction, both $g^j_{i,y}(x)$ and $g^i_{i,y}(y) \in c_i$, and $g^i_{i,y}(y)$ is the $R_@$-fixed point of c_i. Hence $F(\langle\langle i,y\rangle,\langle j,x\rangle\rangle) R_@ F(\langle\langle i,y\rangle,\langle i,y\rangle\rangle)$.

@, Back: we want to show that if $F(\langle\langle i,y\rangle,\langle j,x\rangle\rangle) R_@ v'$, then there is some v such that $F(v) = v'$ and $\langle\langle i,y\rangle,\langle j,x\rangle\rangle R^X_@ v$. By construction, a witness for this existential is $\langle\langle i,y\rangle,\langle i,y\rangle\rangle$.

□

Theorem 4. B2D *is sound and complete for* M.

Proof. From Theorems 1-3. □

C Looking Ahead

The ■ operator is an S5 modal operator, and B2D is a conservative extension of a simple S5 modal logic. In looking ahead to a two-dimensional treatment of other logics, we should ask how two dimensions might be leveraged to upgrade a class of traditional Kripke models whose original accessibility relation(s) capture features of various "flavors" of possibility and necessity.

For example, there is a sense in which the ■ of B2D models an all-inclusive *global* modality, where every world is accessible from every other. This implied globality is why the clause for $\blacksquare\phi$ in §A required no R-restriction. But the accessibility relation for ■ in B2D is *not*, of course, global on the space of world-pairs: for A to remain rigid in the scope of ■, as (@$5_\blacksquare$) mandates, ■ *cannot* shift the world-as-actual parameter. Hence whether the two-dimensional entry (■) expresses a truly *global* modality, or not, depends on which space—the space of worlds, or the space of world-pairs—one takes to be fundamental to the model.

An *Åqvist logic* (Åqvist, 1973; Segerberg, 1973, pg. 77) can be seen as taking the space of worlds to be fundamental, in the following sense. It interprets the *atoms* of $\mathcal{L}$ as functions from worlds to truth-values that are constant in the world-as-actual parameter—that is, as *propositions* in Stalnaker's sense.

The class *At* of atomics thus exhibits special features that arbitrary wffs in the full language do not have. Atoms satisfy the restricted schema (#15):

Definition 6 (Segerberg's Axiom (#15))**.**

(#15) $\alpha \equiv \dagger\alpha$, for $\alpha \in At$.

Call the constraint imposed by (#15) *barcodeness*: if $f_\dagger(w) = f_\dagger(w')$, then for atomic α: $w \vDash \alpha$ iff $w' \vDash \alpha$. Intuitively, this corresponds to atomic matrices which are their own diagonalizations.

This commitment, at the level of atoms, to the fundamentality of worlds and propositions is attractive in considering the question in this section: the question of how to upgrade a class of non-S5, one-dimensional Kripke models into stereo. On the Åqvist view, a 1D frame is "lifted" into two dimensions—that is, into the space of matrices—in a way that conservatively extends its profile throughout the one-dimensional fragment of the original language, which is then enriched by A and †. A way to do this is to duplicate any one-dimensional $R_\triangledown$ relation in the $W \times W$ plane, preserving its frame conditions across the horizontal, and to incorporate axiom schema (#15) into the relevant proof system, thus disallowing non-barcode atoms. The confinement of $R_\triangledown$ to the horizontal respects (@5$_\blacklozenge$), the inability of box-diamond modalities to shift the world-as-actual parameter. (#15) limits the expressive power of the propositional fragment to the expressive power of an arbitrary valuation function paired with the original, pre-lifted frame. (#15)'s limitation on closure under substitution thus follows the spirit of e.g. Williamson (2013), who remarks that "the obvious rationale for insisting on …closure…under uniform substitution in a propositional system is a reading of non-logical sentence letters as *propositional* variables."[12]

Call such an enriched model a "2D lift" of a one-dimensional Kripke frame, and call the lifted frame enriched with @ and † relations a "Matrix lift" of the same frame. Here, we demonstrate that adding (#15) to our B2D completeness result fits the Åqvist mold for simple S5 frames.

Definition 7. Let B2D + (#15) be the proof system axiomatized by Modus Ponens, Necessitation for each primitive modal operator, Axiom (#15), and the axiom schema

(+B2D) ϕ, where ϕ is a theorem of B2D.

Note that B2D + (#15) lacks the rule of Uniform Substitution.[13]

Definition 8 ($\mathsf{F}^\uparrow$, the matrix lift of a frame class F)**.** Where F is a class of Kripke frames $\langle W, \{R_{\triangledown_i} : i \in I\}\rangle$, the matrix lift of F, $\mathsf{F}^\uparrow$, is the class of matrix models $M^X = \langle W^X, \{R_{\triangledown_i} : i \in I\}, V^X\rangle$ s.t.

- $W^X = W \times W$
- $R^X_{\triangledown_i} = \{\langle\langle y, w\rangle, \langle y, w'\rangle\rangle \in W^X \times W^X : wR_{\triangledown_i}w'\}$
- $R^X_{@} = \{\langle\langle y, w\rangle, \langle y', w'\rangle\rangle \in W^X \times W^X : y = y' \text{ and } y' = w'\}$
- $R^X_{\dagger} = \{\langle\langle y, w\rangle, \langle y', w'\rangle\rangle \in W^X \times W^X : w = w' \text{ and } y' = w'\}$
- $V^X : At \mapsto \wp(W^X)$ is such that $\langle y, w\rangle \in V^X(p)$ iff $\langle y', w\rangle \in V^X(p)$

Notation: call such a valuation function V^X a "vertically constant valuation function".

Theorem 5. B2D + (#15) *is sound and complete for* $\mathsf{S5}^\uparrow$*, the matrix lift of the class of S5 Kripke frames.*

Proof. Soundness: For (+B2D), since B2D is valid over M and $\mathsf{S5}^\uparrow \subset \mathsf{M}$, every theorem of B2D is valid over $\mathsf{S5}^\uparrow$. For (#15), suppose for some atom a that $a \equiv \dagger a$ is false at some

[12] Williamson (2013, pg. 76, emphasis added); see also Burgess (1999, pg. 176).
[13] For discussion, see, *inter alia*, Smiley (1982); Holliday et al. (2013).

$\langle y, x\rangle \in W_{\mathcal{M}}$, $\mathcal{M} \in \mathsf{S5}^{\uparrow}$. Then either $\mathcal{M}, y, x \vDash a$ but $\mathcal{M}, x, x \nvDash a$, or $\mathcal{M}, y, x \nvDash a$ but $\mathcal{M}, x, x \vDash a$. Either way, $\mathcal{M}$ is not a matrix lifted Kripke frame.

As for the rules: the proof that Modus Ponens preserves validity is straightforward. For Necessitation, suppose that ϕ is valid over $\mathsf{S5}^{\uparrow}$. Let $\mathcal{M} \in \mathsf{S5}^{\uparrow}$ and let $y, x \in W$. Then since ϕ is valid over $\mathsf{S5}^{\uparrow}$, that means for any $x' \in W$: $\mathcal{M}, y, x' \vDash \phi$, so (i) $\mathcal{M}, y, x \vDash \blacksquare\phi$; (ii) $\mathcal{M}, y, y \vDash \phi$, so $\mathcal{M}, y, x \vDash @\phi$; and (iii) $\mathcal{M}, x, x \vDash \phi$, so $\mathcal{M}, y, x \vDash \dagger\phi$. Since y, x were arbitrary, it follows that $\blacksquare\phi$, $@\phi$, and $\dagger\phi$ are all valid over $\mathsf{S5}^{\uparrow}$. So all the rules preserve validity.

Completeness: suppose Γ is a $\mathsf{B2D}$ + $(\#15)$-consistent set of formulas. Take $\Delta \supseteq \Gamma$ such that Δ is maximally $\mathsf{B2D}$ + $(\#15)$-consistent. By Theorem 4, there is a matrix model $\mathcal{M}$ and some $y, x \in W$ such that $\mathcal{M}, y, x \vDash \Delta$. By Necessitation for $\blacksquare$ and $\dagger$, $\blacksquare \dagger \blacksquare(\alpha \equiv \dagger\alpha) \in \Delta$. Hence, $\mathcal{M}, y, x \vDash \blacksquare \dagger \blacksquare(\alpha \equiv \dagger\alpha)$, which means for any $z, w \in W$: $\mathcal{M}, z, w \vDash \alpha \equiv \dagger\alpha$. So for any $z, z', w \in W$:

$$\begin{aligned}
\langle z, w\rangle \in V^X(p) \quad &\text{iff} \quad \mathcal{M} z, w \vDash p \\
&\text{iff} \quad \mathcal{M}, z, w \vDash \dagger p \\
&\text{iff} \quad \mathcal{M}, w, w \vDash p \\
&\text{iff} \quad \mathcal{M}, z', w \vDash \dagger p \\
&\text{iff} \quad \mathcal{M}, z', w \vDash p \\
&\text{iff} \quad \langle z', w\rangle \in V^X(p).
\end{aligned}$$

Hence, $\mathcal{M} \in \mathsf{S5}^{\uparrow}$.

□

References

Ahmed, Arif (2014). *Evidence, Decision and Causality.* Cambridge University Press.

Åqvist, Lennart (1973). "Modal Logic with Subjunctive Conditionals and Dispositional Predicates." *Journal of Philosophical Logic*, 2(1): pp. 1–76.

Blackburn, Patrick, Maarten de Rijke, and Yde Venema (2002). *Modal Logic*. Cambridge University Press.

Blackburn, Patrick, and Maarten Marx (2002). "Remarks on Gregory's "Actually" Operator." *Journal of Philosophical Logic*, 31: pp. 281–288.

Burgess, John (1999). "Which modal logic is the right one?" *Notre Dame Journal of Formal Logic*, 40: pp. 81–93.

Chalmers, David (2004). "Epistemic two-dimensional semantics." *Philosophical Studies*, 118: pp. 153–226.

Cresswell, Max (2012). *Entities and Indices*, vol. 41. Springer Science & Business Media.

Crossley, John, and Lloyd Humberstone (1977). "The Logic of 'actually'." *Reports on Mathematical Logic*, 8: pp. 11–29.

Davies, Martin, and Lloyd Humberstone (1980). "Two Notions of Necessity." *Philosophical Studies*, 38(1): pp. 1–30.

Elga, Adam (2000). "Self-locating belief and the Sleeping Beauty problem." *Analysis*, 60: pp. 143–147.

Fritz, Peter (2013). "A Logic for Epistemic Two-Dimensional Semantics." *Synthese*, 190: pp. 1753–1770.

Fritz, Peter (2014). "What is the Correct Logic of Necessity, Actuality and Apriority?" *Review of Symbolic Logic*, 7(3): pp. 385–414.

Fusco, Melissa (2018). "Epistemic Time-Bias in Newcomb's Problem." In A. Ahmed (ed.) *Newcomb's Problem*, Cambridge University Press.

Fusco, Melissa (2019). "Naturalizing Deontic Logic: Indeterminacy, Diagonalization, and Self-Affirmation." *Philosophical Perspectives*, 32: pp. 165–187.

Gregory, Dominic (2001). "Completeness and Decidability Results for Some Propositional Modal Logics Containing "Actually" Operators." *Journal of Philosophical Logic*, 30(1): pp. 57–78.

Hawthorne, John, and Ofra Magidor (2009). "Assertion, Context, and Epistemic Accessibility." *Mind*, 118(470).

Hazen, Allen, Benjamin Rin, and Kai Wehmeier (2013). "Actuality in Propositional Modal Logic." *Studia Logica*, 101: pp. 487–503.

Heim, Irene (1982). *The Semantics of Definite and Indefinite Noun Phrases.* Ph.D. thesis, University of Massachusetts, Amherst.

Holliday, Wesley, Thomas Icard, and Tomohiro Hoshi (2013). "Information Dynamics and Uniform Substitution." *Synthese*, 190(Suppl 1): pp. 31–55.

Humberstone, Lloyd (1981). "Relative Necessity Revisited." *Reports on Mathematical Logic*, 13: pp. 33–42.

Kaplan, David (1989). "Demonstratives." In J. Almog, J. Perry, and H. Wettstein (eds.) *Themes from Kaplan*, Oxford University Press.

Kripke, Saul (1980). *Naming and Necessity.* Harvard University Press.

Lampert, Fabio (2018). "Actuality, Tableaux, and Two-Dimensional Modal Logics." *Erkenntnis*, 83: pp. 403–433.

MacFarlane, John (2009). "Nonindexical Contextualism." *Synthese*, 166: pp. 231–50.

Magidor, Ofra (2010). "Review of Our Knowledge of the Internal World by Robert Stalnaker." *Philosophical Review*, 119(3): pp. 384–391.

Restall, Greg (2012). "A cut-free sequent system for two-dimensional modal logic, and why it matters." *Annals of Pure and Applied Logic*, 163: pp. 1611–1623.

Segerberg, Krister (1973). "Two-Dimensional Modal Logic." *Journal of Philosophical Logic*, 2(1): pp. 77–96.

Sider, Theodore (2010). *Logic for Philosophy.* Oxford University Press.

Smiley, Timothy (1982). "The Schematic Fallacy." *Proceedings of the Aristotelian Society*, 83: pp. 1–17.

Stalnaker, Robert (1978). "Assertion." In *Context and Content*, Oxford University Press, pp. 78–95. In Stalnaker (1999).

Stalnaker, Robert (1999). *Context and Content.* Oxford, Oxford University Press.

Stalnaker, Robert (2002). "Common Ground." *Linguistics and Philosophy*, 25(5-6): pp. 701–721.

Stalnaker, Robert (2010). *Our Knowledge of the Internal World.* Oxford University Press.

Thomason, Richmond (1970). "Indeterminist Time and Truth-Value Gaps." *Theoria*, pp. 264–281.

Williamson, Timothy (2013). *Modal Logic as Metaphysics.* Oxford University Press.

Wittgenstein, Ludwig (1953). *Philosophical Investigations.* Wiley-Blackwell.

Wittgenstein, Ludwig (1974). *Tractatus Logico-Philosophicus.* London: Routledge.