

A Two-Dimensional Logic for Two Paradoxes of Deontic Modality

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Abstract

In this paper, we axiomatize the deontic logic in [Fusco 2015](#), which uses a Stalnaker-inspired account of diagonal acceptance and a two-dimensional account of disjunction to treat Ross's Paradox and the Puzzle of Free Choice Permission. On this account, disjunction-involving validities are a priori rather than necessary. We show how to axiomatize two-dimensional disjunction so that the introduction/elimination rules for boolean disjunction can be viewed as one-dimensional projections of more general two-dimensional rules. These completeness results help make explicit the restrictions Fusco's account must place on free-choice inferences. They are also of independent interest, as they raise difficult questions about how to 'lift' a Kripke frame for a one-dimensional modal logic into two dimensions.

1 Introduction

The validity of `OR`-introduction seems to be a basic rule of natural language disjunction: from ϕ , one may infer $\phi \text{ OR } \psi$.¹

Or-Intro. $\phi \models (\phi \text{ OR } \psi)$

Yet the rule apparently fails in the scope of deontic modals. For example, the nonentailment known as Ross's Puzzle illustrates the failure of **Or-Intro** in the scope of 'ought' (henceforth O) ([Ross, 1941](#)):

Ross's Puzzle. $O\phi \not\models O(\phi \text{ OR } \psi)$

As an illustration, observe that (1-a), famously, does not seem to entail (1-b):

¹Throughout, we use ' \models ' for the intuitive entailment. Subscripted turnstiles will denote entailment relative to a specific semantics.

- (1) a. You ought to post the letter.
- b. You ought to post the letter or burn it.

Similarly, **Or-Intro** does not seem valid in the scope of ‘may’ (henceforth *M*). For instance, (2-a) does not seem to entail (2-b):

- (2) a. You may post the letter.
- b. You may post the letter or burn it.

An attractive explanation for this is that (2-b) seems to entail that both disjuncts are permissible—an observation known as Free Choice Permission (Kamp, 1973):

$$\text{FC. } M(\phi \text{ OR } \psi) \models M\phi \wedge M\psi$$

For example, the inference from (3-a) to (3-b) and (3-c) seems valid:

- (3) a. You may have an apple or a pear.
- b. You may have an apple.
- c. You may have a pear.

This would explain why (2-a) doesn’t entail (2-b): if it did, then (2-a) would entail you may burn the letter, which is not the case!

Two caveats about **FC**. First, while (3-a) seems to entail (3-b) and (3-c), it resoundingly fails to entail (4):

- (4) You may have an apple and a pear.

This additional datum is sometimes called ‘exclusivity’.²

$$\text{EX. } M(\phi \text{ OR } \psi) \not\models M(\phi \wedge \psi)$$

Second, the **FC** inference is licensed for both disjuncts only when it’s possible for the agent to make each disjunct true without the other. For instance, suppose you are told you may take an apple or a pear, but it turns out (say, for practical reasons) that you can only take the pear if you also take the apple. In that case, it seems as though you are not, in general, permitted to take the pear. In light of this, the form of free choice we will be interested in makes explicit that it’s possible for each disjunct to be true without the other:³

²See, *inter alia*, Simons (2005), Fox (2007), and Barker (2010).

³Empirical data suggests these two caveats are connected, especially in the case where there are more than two disjuncts. In the simplest case, suppose that for the disjunction $\lceil p_1 \text{ OR } p_2 \text{ OR } \dots \text{ OR } p_n \rceil$ it holds that $\Box(p_i \supset p_{i+1})$. Then it is impossible to make p_1 true without making $\bigwedge_i p_i$ true. In this case, **FC**, but not **FCP**, would hold that $M(p_1 \text{ OR } \dots \text{ OR } p_n)$ entails $M(\bigwedge_i p_i)$, contrary to **EX**. Recent experimental work suggests this entailment is not licensed in these cases (Fusco, ms).

FCP. $M(\phi \text{ OR } \psi), \diamond(\phi \wedge \neg\psi), \diamond(\psi \wedge \neg\phi) \models M\phi \wedge M\psi$

It is well-known that **Ross's Puzzle** and **FCP** cause trouble for standard deontic logic, which interprets O and M as normal modal operators. Normal operators are monotonic: if Δ is a normal modal operator and ϕ entails ψ , then $\Delta\phi$ entails $\Delta\psi$. This means that standard deontic logic predicts that (i) $O\phi$ entails $O(\phi \text{ OR } \psi)$ and $M\phi$ entails $M(\phi \text{ OR } \psi)$, and (ii) if **FCP** holds, then $M\phi$ entails $M\psi$ for any ϕ and ψ . These predictions reveal a deep problem at the foundations of standard deontic logic.

Ross's Puzzle is problematic for standard deontic logic because of two key principles it validates:

Necessitation. If $\models \phi$, then $\models O\phi$

K Axiom. $\models O(\phi \rightarrow \psi) \rightarrow (O\phi \rightarrow O\psi)$

From these principles, we can derive $O\phi \rightarrow O(\phi \text{ OR } \psi)$. Here is the proof:

1. $\phi \rightarrow (\phi \text{ OR } \psi)$ tautology
2. $O(\phi \rightarrow (\phi \text{ OR } \psi))$ **Necessitation**, 1
3. $O(\phi \rightarrow (\phi \text{ OR } \psi)) \rightarrow (O\phi \rightarrow O(\phi \text{ OR } \psi))$ **K Axiom**
4. $O\phi \rightarrow O(\phi \text{ OR } \psi)$ modus ponens, 2, 3

Thus, if one wishes to capture **Ross's Puzzle** in a deontic logic, one seems committed either to rejecting **Necessitation** or rejecting the **K Axiom**. But which is the culprit, and why?

[Fusco \(2015\)](#) suggests that the culprit is **Necessitation**. To see why, let's consider why the inference from (1-a) to (1-b) seems bad. One reason it seems bad to say you ought to post the letter or burn it is that this seems to imply that both disjuncts are permissible. The following inference pattern—Free Choice Obligation—seems intuitively valid:

FCO. $O(\phi \text{ OR } \psi) \models M\phi \wedge M\psi$

Thus, to illustrate, (5-a) seems to entail (5-b) and (5-c):

- (5) a. You ought to either do the dishes or take out the trash.
- b. You may do the dishes.
- c. You may take out the trash.

Moreover, **FCO** follows from **FCP** together with a relatively plausible principle relating 'ought' and 'may':

Ought Implies May. $O\phi \models M\phi$

But if **FCO** holds, then already **Necessitation** is enough to wreak havoc: **Necessitation** plus **FCO** entail that everything is permissible:⁴

- | | |
|-----------------------------------|--------------------------|
| 1. $(\phi \text{ OR } \neg\phi)$ | tautology |
| 2. $O(\phi \text{ OR } \neg\phi)$ | Necessitation , 1 |
| 3. $M\phi \wedge M\neg\phi$ | FCO , 2 |

This provides one motivation for thinking that the way forward for deontic logic is to reject **Necessitation**.

Two-dimensional semantics offers an elegant way of developing **Necessitation**-free modal logics: while (6-a) below is “necessary” in the sense of being knowable a priori, it is not metaphysically necessary, in that things could have been different from how they actually are (Crossley and Humberstone, 1977; Davies and Humberstone, 1980; Kaplan, 1989).

- (6) a. Everything is as it actually is.
 b. Necessarily, everything is as it actually is.

Perhaps, then, the key to solving these puzzles is to move to a two-dimensional framework. This is the strategy pursued by Fusco (2015).

The main idea behind Fusco’s semantics is to introduce a two-dimensional entry for ‘OR’ on which it behaves classically in unembedded contexts but nonclassically under the scope of deontic modals. However, while this semantics captures **Ross’s Puzzle** and **FCP**, there still remains a further question regarding what the complete logic of her semantics is.

In this paper, we answer this question. We show how to axiomatize the logic of two-dimensional disjunction so that the introduction/elimination rules for boolean disjunction can be viewed as one-dimensional projections of more general rules. In addition, we prove several soundness and completeness results for two-dimensional deontic logics, which help make explicit the background assumptions and scope of Fusco’s account.

We take these completeness results to be of independent interest, as they raise interesting questions about how to “lift” a Kripke frame for a one-dimensional modal logic into two dimensions. These issues arise especially in the context of the picture of communication from Stalnaker 1978, where diagonalization plays a key role. In particular, two-dimensionalizing deontic logic in a Stalnakerian framework seems to require an interesting and substantive metaphysical assumption about the nature of deontic accessibility, viz., that which worlds are deontically accessible does not vary from world-to-world.⁵

⁴Note it does not help here to restrict **FCO** to ϕ and ψ that are possible; that still would imply that for any contingent ϕ is permissible.

⁵This bears resemblance to a point made by Hawthorne and Magidor (2009) about epistemic accessibility.

Here is a brief outline. In § 2, we review the philosophical motivations, formal semantics, and logic of standard two-dimensionalism. In § 3, we present Fusco’s two-dimensional semantics for disjunction and articulate different ways to axiomatize the logic of disjunction prior to adding deontic modals. In § 4, we extend two-dimensionalism with deontic modals and show how doing so can capture both **Ross’s Puzzle** and **FCP**. The details of the completeness proofs are given in the technical appendices.

2 A Crash Course in Two-Dimensionalism

2.1 Two Notions of Necessity and Two Notions of Consequence

In two-dimensional semantics, the truth of a well-formed formula ϕ in a model \mathcal{M} is relativized to two parameters. The first, the *world-as-actual*, plays a role in determining the propositional content expressed by ϕ .⁶ The second, the *world of evaluation*, plays the role of the world in which that proposition’s truth-value is interrogated; it is also the only parameter shifted by the standard modal operators \Box and \Diamond . Two basic two-dimensional operators, $@$ (usually glossed as ‘actually’) and \dagger , can be added alongside \Box and \Diamond :⁷

(\Box) $y, x \Vdash \Box\phi$ iff for all x' : $y, x' \Vdash \phi$

($@$) $y, x \Vdash @\phi$ iff $y, y \Vdash \phi$

(\dagger) $y, x \Vdash \dagger\phi$ iff $x, x \Vdash \phi$

One motivation for moving to a two-dimensional framework is the fact that we can regiment several philosophically important distinctions in a unified way (Davies and Humberstone, 1980). To illustrate, recall (6-a) and (6-b) from § 1:

- (6) a. Everything is as it actually is.
 b. Necessarily, everything is as it actually is.

We observed that while (6-a) seems to be a logical truth, it does not seem to be necessarily true, i.e., (6-b) seems false. The problem is that we cannot accept (6-a) as a logical truth in a normal modal logic while denying (6-b). Let \boxtimes stand

⁶We follow the literature here in defining a proposition to be a function from worlds to truth-values.

⁷Technically, the satisfaction relation \Vdash should always be relativized to a model \mathcal{M} . In our informal exposition, however, we will often drop mention of the model for readability.

for \lceil Everything is as it actually is \rceil .⁸ In the two-dimensional framework, we can analyze this sentence as follows:⁹

(\boxtimes) $y, x \Vdash \boxtimes$ iff $y = x$

Then we have the following argument for (6-b):

1. \boxtimes is a logical truth
2. **Necessitation** holds, i.e., if ϕ is a logical truth, then so is $\Box\phi$.
3. $\Box\boxtimes$ is a logical truth.

If we want to deny the conclusion, we need to reject one of the premises. But which?

Two-dimensionalism provides us with an elegant answer: which premise you should give up depends on your notion of consequence. First, let us say an argument is **strictly valid** just in case it preserves truth over all points of evaluation, i.e., all pairs of worlds $\langle y, x \rangle$. Strict validity captures the thought that a valid argument is meant to preserve truth *in every possible scenario*. On this notion of logical consequence, **Necessitation** still holds: if ϕ is true at every point of evaluation $\langle y, x \rangle$, then so is $\Box\phi$. However, the first premise of the argument for (6-b) is false, since there are non-diagonal points where the x - and y -coordinates do not agree on what is the case—that is, whatever world is actual, it still *could* have been that things were different from how they actually are. Since \boxtimes only holds on diagonal points, it is not a strict logical truth.

However, \boxtimes does hold on all *diagonal* points. An argument is **diagonally valid** just in case it preserves truth over all diagonal points, i.e., all pairs of the form $\langle z, z \rangle$. This captures the thought that a valid argument is meant to preserve truth *no matter which world is actual*. On this notion of logical consequence, the first premise of the argument for (6-b) is true, but the second premise is false—e.g., \boxtimes is diagonally valid, but $\Box\boxtimes$ is not. In general, whenever something is diagonally but not strictly valid, we'll have a counterexample to **Necessitation**.

Quantification over diagonal points also allows one to represent the difference between *metaphysical necessity* and *a priority* in a two-dimensional framework. This is just the difference between ϕ 's holding everywhere along the x -axis and ϕ 's being diagonally valid.¹⁰ That is, fixing y as the actual world, ϕ is metaphysically

⁸For similar sentential constants, compare (for “floating” actuality) ‘I’ from Humberstone (2004, pg. 54), and (for “anchored” actuality) the constant ‘ n ’ in Meredith and Prior (1965). In a language with propositional quantifiers and @, we could express \boxtimes as $\forall p(@p \rightarrow p)$.

⁹These truth conditions ignore complications that arise when x and y are qualitative duplicate worlds. We set this complication aside.

¹⁰See Davies and Humberstone 1980; Kaplan 1989. For classic examples where metaphysical necessity comes apart from a priority, see Kripke 1980; Kaplan 1989.

necessary iff it holds at $\langle y, x \rangle$ for all x , whereas ϕ is a priori iff it holds at $\langle z, z \rangle$ for all z .¹¹ The latter notion can be represented by introducing a new defined operator for a *a priori truth*: $A\phi := \Box\uparrow\phi$. It is easy to check that A has the following truth conditions:

(A) $y, x \Vdash A\phi$ iff for all $z: z, z \Vdash \phi$.

While \boxtimes ('everything is as it actually is') is not metaphysically necessary, it is a priori: that is, $A \boxtimes$ is a strict logical truth. In fact, even though **Necessitation** does not hold for diagonal validity, the following variant of **Necessitation** does:

A Priorization. If $\models \phi$, then $\models A\phi$.

That is: if ϕ is a diagonal logical truth, then so is $A\phi$. Strict validity is a formalization of necessary entailment, while diagonal validity is a formalization of a priori entailment.

2.2 Stalnaker's Two Dimensionalism

Another classic application of two-dimensionalism, which will provide us with a metalanguage (rather than object language) gloss on A , comes from [Stalnaker 1978](#). There, Stalnaker uses two-dimensionalism to model conversations in which speakers informatively assert sentences whose propositional content is under threat from indeterminacy. To clarify the problem, he introduces the notion of a *context set*: a set of possible worlds left open by what is accepted in common by the interlocutors of a conversation. *Conversational update* on an assertion of ϕ is a quasi-Bayesian procedure which aims to eliminate from the context set any worlds which are incompatible with ϕ 's content.

Suppose, however, that it is unknown—or perhaps, in the case of the open future, even metaphysically indeterminate—which world is actual. Then there will be cases in which it is unknown or indeterminate what the content of an assertion of ϕ is, and thus unclear which worlds should be “thrown out” of the context set if the assertion is to be accepted. Stalnaker's flagship example involves the word ‘you’ in the asserted sentence ‘You are a fool’ (pg. 81). If it is unknown, or indeterminate, who the addressee of the context is, it will be unknown, or indeterminate, which person is being claimed to be a fool, and so it will be unknown or indeterminate which update is being proposed.

Following Stalnaker, we can visualize the situation using two-dimensional matrices. In these matrices, the rows represent the world-as-actual (y -coordinate)—a role which contributes to determining the proposition expressed by a sentence

¹¹This idea can be found, *inter alia*, in [van Fraassen \(1977\)](#) and [Lewis \(1973, §2.8\)](#), who in turn cites [Kamp \(1971\)](#) and [Vlach \(1973\)](#). See also [Chalmers \(2004\)](#).

(e.g., determining the referent of ‘you’)—while the columns represent the world of evaluation (x -coordinate), where the content, once determined, is assessed. If the content of a sentence depends on which world is actual, then the rows of the matrix will not be identical.

So to use Stalnaker’s example, suppose the speaker is addressing O’Leary in worlds i and j and to Daniels in world k , and moreover O’Leary is a fool in worlds i and k but not in world j , whereas Daniels is a fool only in j . In that case, we can represent the distribution of truth values of ‘You are a fool’ as in Figure 1. For instance, the cell on row i and column j contains the truth value of ‘You are a fool’ at the point $\langle i, j \rangle$, which is true iff the person that the speaker is addressing in world i is a fool in world j . Since at world i , the speaker is addressing O’Leary, and since O’Leary is not a fool at world j , ‘You are a fool’ is false at $\langle i, j \rangle$.

	i	j	k
i	T	F	T
j	T	F	T
k	F	T	F

‘You are a fool’

Figure 1: A two-dimensional matrix for ‘You are a fool’ (Stalnaker, 1978, pg. 81).

Which worlds need to be thrown out of the context set upon accepting an assertion of ‘You are a fool’ depends on which world is actual: if i or j is actual, we throw out j ; if k is actual, we throw out i and k . But—to reiterate the problem Stalnaker is raising—which world is actual is exactly what is either unknown or indeterminate in context (*op. cit.*, pg. 90).

In such a predicament, Stalnaker proposes that a sentence ϕ can be rationally (re)interpreted as $\dagger\phi$. As Stalnaker puts it, this move, called **diagonalization**, interprets ϕ as something like “what ϕ says is true”, where “what ϕ says” is the proposition expressed by ϕ given a world as actual (pg. 82).¹² This is visualized in Figure 2. Diagonalization restores a **uniformity** condition to matrices: the same proposition is expressed relative to each candidate for actuality.¹³ In a scenario illustrated by Figure 1, updating the context set with the diagonalized assertion amounts to eliminating worlds j and k from the context set.

¹²For a suggestion that diagonalization can also occur in the scope of attitude verbs like ‘believes’, see Stalnaker (1981).

¹³In the literature, Hawthorne and Magidor (2009) object that diagonalization does not restore uniformity unless certain tendentious conditions on knowledge are met: namely, that the same worlds horizontally accessible on each row. We bracket this concern here, particularly because our discussion is applicable to nonepistemic as well as epistemic indeterminacy.

	<i>i</i>	<i>j</i>	<i>k</i>
<i>i</i>	T	F	T
<i>j</i>	T	F	T
<i>k</i>	F	T	F

⌈You are a fool⌋

	<i>i</i>	<i>j</i>	<i>k</i>
<i>i</i>	T	F	F
<i>j</i>	T	F	F
<i>k</i>	T	F	F

⌈†You are a fool⌋

Figure 2: Visualizing diagonalization using matrices.

Stalnaker’s theory of assertion reveals a further distinction in the notion of logical consequence worth highlighting. In § 2.1, we introduced a **classical** notion of consequence, which defines validity in terms of preservation of *truth*: an argument is valid in this sense if the conclusion is true at any point of evaluation where the premises are true. But there is also an **informational** notion of consequence, which defines validity in terms of preservation of *acceptance*. An argument is valid in this sense if the conclusion is *accepted* at any context set where the premises are accepted.¹⁴ In a static one-dimensional framework, acceptance can be modeled as global truth, i.e., truth relative to every world in a context set. Thus, Γ informationally entails ϕ whenever $\Box\Gamma := \{\Box\gamma \mid \gamma \in \Gamma\}$ classically entails $\Box\phi$.¹⁵

$$\begin{aligned} \Gamma \text{ classically entails } \phi &:= \forall x: (x \Vdash \Gamma \Rightarrow x \Vdash \phi) \\ \Gamma \text{ informationally entails } \phi &:= (\forall x: x \Vdash \Gamma) \Rightarrow (\forall x: x \Vdash \phi) \end{aligned}$$

But what about diagonalization? In Stalnaker’s two-dimensional framework, diagonalization may occur before acceptance into the common ground.¹⁶ Hence there is a strict-diagonal distinction to be had at the level of informational consequence as well as classical consequence. **Informational strict consequence** can be glossed like this: no matter which world is actual, if the content of the premises were accepted, so would the content of the conclusion. Stated in terms of classical strict entailment: Γ informationally strictly entails ϕ whenever $\Box\Gamma$ classically strictly entails $\Box\phi$. **Informational diagonal consequence**, by contrast, can be glossed like this: if the diagonalized content of the premises are accepted, then

¹⁴For more on this notion, see Stalnaker 1975’s notion of reasonable inference, as well as, *inter alia*, Veltman 1996; Yalcin 2007; Bledin 2015.

¹⁵The difference between classical and informational consequence mirrors the distinction between *local* and *global* consequence that one finds in the literature on modal logic (Blackburn et al., 2002). For unarticulated ‘boxlike’ modalities in natural language, it is worth noting Kratzer’s influential proposal that bare indicative conditionals contain a covert necessity operator in the consequent Kratzer (1981, 1986).

¹⁶Indeed, Heim (2004, §9) suggests *always* diagonalizing, noting that diagonalization is superfluous if uniformity is already satisfied.

so is the diagonalized content of the conclusion. Stated in terms of classical strict entailment: Γ informationally diagonally entails ϕ whenever $\Box\uparrow\Gamma := \{\Box\uparrow\gamma \mid \gamma \in \Gamma\}$ classically strictly entails $\Box\uparrow\phi$.

Because $A := \Box\uparrow$ can be seen as a priority of the relevant, contextually-situated kind, we will also call informational diagonal consequence **a priori consequence**, which we will write as ' \triangleright '. This notion of consequence matches more closely with the preservation of acceptance in Stalnaker's framework. Moreover, it is a priori consequence which will be key to both validating **FCP** and blocking **Ross's Puzzle** on Fusco's semantics: it is a notion of consequence that is just weak enough to validate the former but not the latter.

Thus, we have four notions of consequence—varying along two dimensions, strict vs. diagonal and classical vs. informational—illustrated in Figure 3.¹⁷ A priori consequence occupies the lower right-hand side of the relevant space of possibilities.

	classical	informational
strict	$\Gamma \models \phi$	$\Box\Gamma \models \Box\phi$
diagonal	$\uparrow\Gamma \models \uparrow\phi$	$\Box\uparrow\Gamma \models \Box\uparrow\phi$ $\Gamma \triangleright \phi$

Figure 3: Different notions of consequence summarized.

2.3 Axiomatization of Two-Dimensional Semantics

One of our primary goals in this paper is to give an axiomatization of Fusco's two-dimensional extension of deontic logic. So we should first review axiomatizations for ordinary two-dimensional logic before extending them with deontic modals. We give axiomatizations of classical strict consequence, as the other notions of consequence can be defined in terms of classical strict consequence using \uparrow and \Box (as in Figure 3).

¹⁷Note that in the two-dimensional setting, *informational* strict consequence does not align neatly with *global* strict consequence. Global consequence is usually defined in terms of points of evaluation: an argument is **globally valid** if its conclusion holds at every point of evaluation whenever its premises hold at every point of evaluation. Stated in terms of classical (i.e., local) strict entailment: Γ globally strictly entails ϕ whenever $\Box\uparrow\Box\Gamma := \{\Box\uparrow\Box\gamma \mid \gamma \in \Gamma\}$ classically strictly entails $\Box\uparrow\Box\phi$. This is not the same as informational strict consequence, since the world-as-actual is held fixed from premises to conclusion. Thus, informational strict consequence is stronger than global strict consequence. On the other hand, informational diagonal consequence and global diagonal consequence do coincide.

Definition 1. Given a set $\text{Prop} = \{p_1, p_2, p_3, \dots\}$ of proposition letters, the basic two-dimensional language \mathcal{L}^{2D} is defined recursively in Backus-Naur form as follows:

$$\phi ::= \text{Prop} \mid \neg\phi \mid (\phi \wedge \phi) \mid \Box\phi \mid @\phi \mid \dagger\phi.$$

$\Diamond, \vee, \rightarrow$ and \leftrightarrow defined as usual; in particular, note that $(\phi \vee \psi) := \neg(\neg\phi \wedge \neg\psi)$. We also define $A\phi$ (it is a priori that ϕ) as $\Box\dagger\phi$.

Definition 2. A **basic matrix frame** is a Kripke frame $\langle W \times W, R_\Box, R_\@, R_\dagger \rangle$, where $W \neq \emptyset$ and:

- $\langle y, x \rangle R_\Box \langle y', x' \rangle$ iff $y = y'$
- $\langle y, x \rangle R_\@ \langle y', x' \rangle$ iff $y' = x' = y$.
- $\langle y, x \rangle R_\dagger \langle y', x' \rangle$ iff $y' = x' = x$.

Let \mathbf{M} be the class of basic matrix frames. A **basic matrix model** is a pair of a basic matrix frame $\langle W \times W, R_\Box, R_\@, R_\dagger \rangle$ and a valuation function $V: \text{Prop} \rightarrow \wp(W \times W)$.

We define the satisfaction relation \models between a pointed matrix model \mathcal{M}, y, x and a \mathcal{L}^{2D} -formula ϕ in the usual manner for the Kripke semantics. Where Γ is a set of formulas, we'll write $\mathcal{M}, y, x \models \Gamma$ to mean that $\mathcal{M}, y, x \models \gamma$ for all $\gamma \in \Gamma$. Finally, we define classical strict consequence: where Γ is a set of \mathcal{L}^{2D} -formulas and ϕ is a \mathcal{L}^{2D} -formula, $\Gamma \models_{2D} \phi$ iff for all pointed basic matrix models \mathcal{M}, y, x , if $\mathcal{M}, y, x \models \Gamma$, then $\mathcal{M}, y, x \models \phi$.

Definition 3. Let **2D** be the Hilbert-style proof system consisting of the following axioms and rules: (i) all propositional tautologies as axioms; (ii) the rules of Modus Ponens and Uniform Substitution; (iii) the **K Axiom** and **Necessitation** for each primitive modal operator; and (iv) the following axioms:

$(T_\Box) \Box p \rightarrow p$	$(R_\dagger) \dagger p \leftrightarrow \neg\dagger\neg p$
$(5_\Box) \Diamond p \rightarrow \Box\Diamond p$	$(X_\rightarrow) \dagger(p \rightarrow @p)$
$(@5_\Diamond) \Diamond @p \rightarrow @p$	$(Y_\rightarrow) @ (p \rightarrow \dagger p)$
$(G) \Box p \rightarrow @p$	$(4_A) \Box\dagger p \rightarrow \Box\dagger\Box\dagger p$
$(R_\@) @p \leftrightarrow \neg @\neg p$	$(5_A) \Diamond\dagger p \rightarrow \Box\dagger\Diamond\dagger p$

We write $\Gamma \vdash_{2D} \phi$ if for some $\gamma_1, \dots, \gamma_n \in \Gamma$, the formula $(\gamma_1 \wedge \dots \wedge \gamma_n) \rightarrow \phi$ is derivable in **2D**.

The following is proven in [Fusco forthcoming](#).

Theorem 1. **2D** is sound and complete for classical strict consequence over \mathbf{M} .

A priori consequence can then just be defined in terms of classical strict consequence: $\Gamma \triangleright_{2D} \phi$ iff for some $\gamma_1, \dots, \gamma_n \in \Gamma: \vdash_{2D} (A\gamma_1 \wedge \dots \wedge A\gamma_n) \rightarrow A\phi$.

2.4 Åqvist logic

The matrix semantics given above is more general than the usual presentation of two-dimensionalism in at least one respect: we allow atomics to represent arbitrary matrices like the one provided for $\lceil \text{You are a fool} \rceil$, whose content varies with the world-as-actual.

By contrast, according to one common strand of two-dimensionalism, valuation functions interpret atomics as sets of *worlds*, not sets of *pairs of worlds*. This means the truth of atomics never depends on the world-as-actual parameter. These “standard” truth conditions build in the requirement that the atomics are reserved for sentences that satisfy the Stalnakerian uniformity condition introduced in § 2.2, i.e., that $\langle y, x \rangle \in V(p)$ iff $\langle y', x \rangle \in V(p)$ for any x, y, y' . For these sentences, necessity and a priority coincide.¹⁸

It is natural to ask how imposing this restriction on the valuations of atomics affects the logic given in § 2.3. We call such logics **Åqvist logics** (Åqvist, 1973; Segerberg, 1973). It turns out that their distinguishing axiom is (Å):

(Å) $\alpha \leftrightarrow \dagger\alpha$ for any atomic α (Segerberg, 1973, pg. 95)

In terms of matrices, (Å) says that rows of the matrix of an atomic must be duplicates, i.e., each column must either be a column of Ts or a column of Fs.

Definition 4. A matrix model $\langle W \times W, R_{\square}, R_{\otimes}, R_{\dagger}, V \rangle$ is **Åqvist** iff for all $x, y, y' \in W$:

$$\langle y, x \rangle \in V(p) \quad \Leftrightarrow \quad \langle y', x \rangle \in V(p).$$

Let $M^{\text{Å}}$ be the class of Åqvist matrix models.

Note that we can define an “Åqvist operator” Å which applies to a formula ϕ iff ϕ ’s truth is insensitive to the world-as-actual parameter, i.e., iff it is interpreted the same way on every row:

$$\text{Å}\phi := \square\dagger\square(\phi \leftrightarrow \dagger\phi)$$

Then we could reformulate (Å) as the axiom schema $\text{Å}\alpha$ for any atomic α . The Å operator turns out to be important for Fusco’s explanation of **FCP**, as the inference only holds in her semantics for ϕ and ψ that have this “uniformity” feature.

As Segerberg is at pains to emphasize, an Åqvist logic is *not*, in general, closed under uniform substitution.¹⁹ Rather, it is closed only under substitution within

¹⁸One can compare here the *PQTI* sentences of Chalmers (2012).

¹⁹For discussion, see, *inter alia*, Smiley 1982; Holliday et al. 2013.

the 1D fragment of the language.²⁰ Thus, in order to axiomatize Åqvist logics, we need to formulate proof systems that are not closed under Uniform Substitution, but still allow Uniform Substitution when no appeal to (\mathring{A}) has been made. This is achieved as follows:

Definition 5. Let $2D\mathring{A}$ be the Hilbert-style proof system consisting of Modus Ponens, Necessitation for each primitive modal operator, (\mathring{A}) , and the axiom schema:

(2D) ϕ , where ϕ is a theorem of $2D$.

In $2D\mathring{A}$, we can appeal to Uniform substitution in deriving an ordinary theorem of $2D$. But once we start reasoning with the consequences of (\mathring{A}) , we are no longer allowed to appeal to Uniform Substitution and must instead rely on the other rules and axioms. The following is proven in [Fusco forthcoming](#):

Theorem 2. $2D\mathring{A}$ is sound and complete for strict classical consequence over MA .

3 Two-Dimensional Disjunction

3.1 An Actuality-sensitive Semantics for Disjunction

We now return to Fusco’s strategy for solving the deontic puzzles sketched in § 1. According to this two-dimensional strategy, what we need for **Ross’s Puzzle** and **FCP** is a semantics for disjunction where **Or-Intro** is a priori, but not necessary. In this section, we review Fusco’s proposal and explain how an axiomatization for the basic two-dimensional language can be extended to include her two-dimensional disjunction.

This semantics is motivated by [Groenendijk and Stokhof \(1982\)](#) and [Lewis \(1982\)](#)’s two-dimensional approach to “whether p or q ” attitude ascriptions, as in

- (7) Bob knows [whether [p OR q]].
- (8) Bob doesn’t know [whether [p OR q]].

The insight here is that (7) and (8) credit Bob with knowledge and ignorance, respectively, of the *actually true* answer to the question $\lceil p, \text{ or } q? \rceil$. According to

²⁰This limitation on closure under substitution follows the spirit of, e.g., [Williamson \(2013\)](#), who remarks that “the obvious rationale for insisting on . . . closure . . . under uniform substitution in a propositional system is a reading of non-logical sentence letters as *propositional variables*” ([Williamson, 2013](#), pg. 76, emphasis added; see also [Burgess, 1999](#), pg. 176). Similarly, in a 2D framework [Gregory \(2001\)](#) remarks that “Good modal [arguments]—at least one variety—are ones which are informally sound...[meaning that] for any interpretation of \mathcal{L} , the *propositions* assigned to [the premises] entail the *proposition* assigned to [the conclusion] (*op. cit.*, pg. 58, emphasis added). This restriction on closure will exclude arbitrary dipropositions, like \bowtie and the (undiagonalized) $\lceil \text{You are a fool} \rceil$, from the relevant standard of good argument.

Groenendijk & Stokhof and Lewis, \lceil whether p or q \rceil is a two-dimensional propositional concept, which expresses the proposition that p , if p rather than q is true in the world-as-actual, and q , if q rather than p is true in the world-as-actual. Thus a speaker of (7) or (8) can credit Bob with knowledge (or ignorance) of this proposition, even if he himself does not know it.

Building off this, Fusco (2015) proposes we interpret *bare* disjunction along similar lines: if p but not q is actually true, then p OR q expresses the proposition that p ; and if q but not p is actually true, then p OR q expresses the proposition that q . The key difference between the semantic entries for \lceil whether p or q \rceil above and Fusco’s semantics for p OR q concerns the cases where either both disjuncts true or neither is: in that case, for Fusco, p OR q just expresses the ordinary boolean disjunction $p \vee q$.²¹

We can state this semantics for OR in terms of a simplified answerhood operator (Dayal, 1996, 2016). This operator takes a world w and disjuncts ϕ and ψ as arguments and outputs the true-in- w answers to the question of whether ϕ or ψ , if there are any. Otherwise, it outputs both answers:

$$Ans(w, \phi_i, \phi_j) = \begin{cases} \{\phi_i\} & \text{if } w, w, \Vdash \phi_i \text{ and } w, w, \nVdash \phi_j \\ \{\phi_i, \phi_j\} & \text{otherwise.} \end{cases}$$

The truth conditions for disjunction can then be given as follows:

$$(OR) \quad y, x \models (\phi \text{ OR } \psi) \text{ iff } \exists \alpha \in Ans(y, \phi, \psi): y, x \Vdash \alpha.$$

Figure 4 illustrates these truth conditions in matrix form. Notice that OR is, as Humberstone (2020, §4.7) puts it, a kind of two-dimensional isotope of boolean \vee —that is, OR and \vee are equivalent *along the diagonal*. Thus, for diagonal consequence, all the standard rules governing boolean disjunction hold of OR. But off the diagonal, OR and \vee come apart. For example, in Figure 4, $w_2, w_3 \nVdash p \text{ OR } q$ even though $w_2, w_3 \Vdash p$.

On this picture of disjunction, $p \text{ OR } \neg p$ is like \bowtie , in that it is an a priori truth that cannot be necessitated. (Indeed, if there are only finitely many atomics p_1, \dots, p_n and all the rows of the truth table are represented by a unique world, then

²¹Groenendijk & Stokhof’s entry *presupposes* that exactly one of $\{p, q\}$ is true in the actual world (*op. cit.* pg. 184), leaving the proposition expressed by the disjunction undefined along the bottom row of the matrix in Figure 4. Lewis’s entry (*op. cit.*, pg. 52) would result in a matrix with F-F-F-F along the same row. But—to preview how the semantics will work under deontic operators like *ought* (O)—this will give the wrong intuitive results for the sentences in **Ross’s Puzzle** and **FCP**. For example, Lewis’s entry under O would entail that (5-a) is equivalent to $\lceil \hat{O} \perp \rceil$ in cases where a lazy agent neither (actually) takes out the trash nor (actually) washes the dishes.

It is also worth noting the great deal of post-Groenendijk and Stokhof 1982 work (not in a two-dimensional tradition) on the general category of concealed questions into which (7) and (8) falls. For developments in treating this class in inquisitive semantics, see, e.g., Roelofsen 2019.

$\bigwedge_i(p_i \text{ OR } \neg p_i)$ generates the same matrix as \boxtimes .) The feature of disjunction will be responsible for nonclassical behavior in the scope of deontic modals (§ 4).

	w_1	w_2	w_3	w_4
w_1	T	T	T	F
w_2	T	T	F	F
w_3	T	F	T	F
w_4	T	T	T	F

$p \text{ OR } q$

	w_1	w_2	w_3	w_4
w_1	T	T	T	F
w_2	T	T	T	F
w_3	T	T	T	F
w_4	T	T	T	F

$p \vee q$

Figure 4: Matrix for $p \text{ OR } q$, compared with that of $p \vee q$, where w_1 is an arbitrary $(p \wedge q)$ -world, w_2 is a $(p \wedge \neg q)$ -world, w_3 is a $(\neg p \wedge q)$ -world, and w_4 is a $(\neg p \wedge \neg q)$ -world.

Note also that $(\phi \text{ OR } \psi)$ will generally not be \mathring{A} qvist, *even if* both ϕ and ψ are. **Figure 5** shows how two-dimensional disjunction, like the @ operator itself, can build non- \mathring{A} qvist matrices out of the matrices for \mathring{A} qvist atomics. The diagonals of these matrices for disjunction reflect the classical profile of \vee : the more disjuncts a classical disjunction has, the more states in a classical (viz., one-dimensional) truth table it is true in.

3.2 Axiomatics

What is the logic of two-dimensional OR like? While some classical inference patterns governing \vee (e.g., **Or-Intro**) are not strictly valid for OR , some are. For instance, the following principles are still classically strictly valid even with Fusco's two-dimensional OR :

Idempotence. $\phi \models (\phi \text{ OR } \phi)$

Commutativity. $\phi \text{ OR } \psi \models \psi \text{ OR } \phi$

Associativity. $\phi \text{ OR } (\psi \text{ OR } \chi) \models (\phi \text{ OR } \psi) \text{ OR } \chi$

Or-Elim. $\phi \text{ OR } \psi, \neg \phi \models \psi$

Thus, the logic for OR is not entirely divorced from the logic of \vee off the diagonal. But it does raise the question of which principles, exactly, are the result of projecting from the two-dimensional OR into a one-dimensional framework. Thus we turn to the question of how to axiomatize our basic two-dimensional language \mathcal{L}^{2D} when extended with OR .

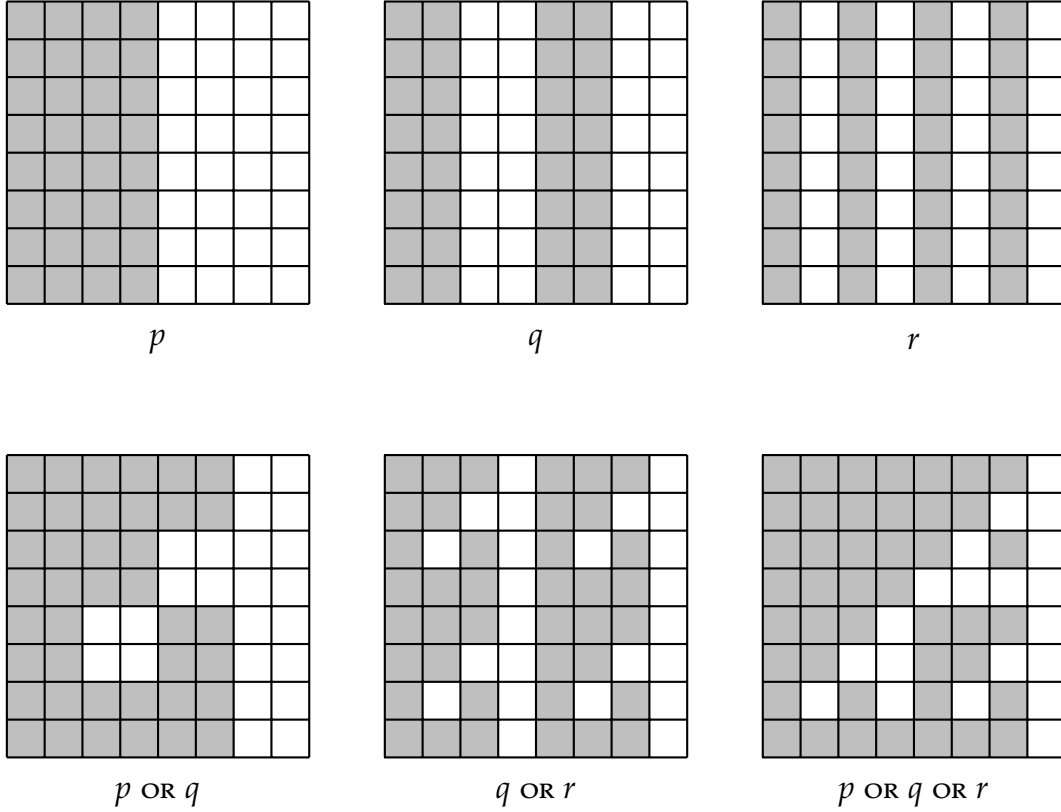


Figure 5: Matrices with 8 worlds per axis. Shading indicates ‘true’; white ‘false’.

In a sense, this task is trivial; as [Humberstone \(2020\)](#) notes, $\phi \text{ OR } \psi$ on Fusco’s semantics can be expressed using @:²²

$$(@(\phi \wedge \neg\psi) \rightarrow \phi) \wedge (@(\neg\phi \wedge \psi) \rightarrow \psi) \wedge (@(\phi \leftrightarrow \psi) \rightarrow (\phi \vee \psi))$$

Thus, we could simply view OR as a defined connective. However, this definition is long-winded; it would be nice to know whether there are more illuminating axioms governing OR directly.

Indeed, there are. Let $\mathcal{L}^{2\text{D}}(\text{OR})$ be the result of extending $\mathcal{L}^{2\text{D}}$ with a primitive OR . Then, as noted above, we could axiomatize 2D extended with OR using the following defining axiom:

$$(\text{OR}_{df}) (p \text{ OR } q) \leftrightarrow [(@ (p \wedge \neg q) \rightarrow p) \wedge (@ (\neg p \wedge q) \rightarrow q) \wedge (@ (p \leftrightarrow q) \rightarrow (p \vee q))]$$

Alternatively, we could replace (OR_{df}) with the following six axioms, which more closely mirror the standard introduction and elimination rules for disjunction:

²²In fact, the opposite is true, too: $@\phi$ can be expressed as $(\phi \text{ OR } \neg\phi) \leftrightarrow \phi$.

- (OR I₁) ($@p \wedge p$) \rightarrow (p OR q)
 (OR I₂) ($@q \wedge q$) \rightarrow (p OR q)
 (OR I₃) ($@\neg(p \vee q) \wedge (p \vee q)$) \rightarrow (p OR q)
 (OR E₁) (p OR q) \rightarrow ($p \vee q$)
 (OR E₂) ($(p$ OR $q) \wedge @\neg p$) \rightarrow ($@\neg q \vee q$)
 (OR E₃) ($(p$ OR $q) \wedge @\neg q$) \rightarrow ($@\neg p \vee p$)

It is straightforward to show that $\mathbf{2D} + (\text{OR}_{df})$ is equivalent to $\mathbf{2D} + (\text{OR I}_1) - (\text{OR E}_3)$. The latter, however, more clearly manifest the way in which the rules governing \vee can be seen as one-dimensional projections from more general two-dimensional rules governing OR. Since $@\phi$ is diagonally equivalent to ϕ , (OR I₁) and (OR I₂) reduce to the standard disjunction introduction rule; (OR I₃) diagonally follows from explosion; (OR E₁) is already an articulation of the disjunction elimination rule in terms of disjunctive syllogism; and (OR E₂) and (OR E₃) diagonally follow from the fact that everything implies a tautology.

Using these axioms, we can directly derive the collapse of OR and \vee along the diagonal. To facilitate the reasoning, observe that the following rule is admissible in $\mathbf{2D}$ for any primitive modal operator Δ :

RK. If $\phi_1, \dots, \phi_n \models \psi$, then $\Delta\phi_1, \dots, \Delta\phi_n \models \Delta\psi$.

Here, then, is an axiomatic proof of $\Box\uparrow((\phi \text{ OR } \psi) \leftrightarrow (\phi \vee \psi))$ —the a priori equivalence of OR and \vee (with instances of Uniform Substitution suppressed):

1. $(\phi \text{ OR } \psi) \rightarrow (\phi \vee \psi)$ (OR E₁)
2. $\uparrow((\phi \text{ OR } \psi) \rightarrow (\phi \vee \psi))$ **Necessitation**, 1
3. $\uparrow(\phi \rightarrow @\phi)$ (X \rightarrow)
4. $\uparrow(\psi \rightarrow @\psi)$ (X \rightarrow)
5. $(@\phi \wedge \phi) \rightarrow (\phi \text{ OR } \psi)$ (OR I₁)
6. $(@\psi \wedge \psi) \rightarrow (\phi \text{ OR } \psi)$ (OR I₂)
7. $\uparrow((@\phi \wedge \phi) \rightarrow (\phi \text{ OR } \psi))$ **Necessitation**, 5
8. $\uparrow((@\psi \wedge \psi) \rightarrow (\phi \text{ OR } \psi))$ **Necessitation**, 6
9. $\uparrow(\phi \rightarrow (\phi \text{ OR } \psi))$ **RK**, 3, 7
10. $\uparrow(\psi \rightarrow (\phi \text{ OR } \psi))$ **RK**, 4, 8
11. $\uparrow((\phi \text{ OR } \psi) \leftrightarrow (\phi \vee \psi))$ **RK**, 2, 9, 10
12. $\Box\uparrow((\phi \text{ OR } \psi) \leftrightarrow (\phi \vee \psi))$ **Necessitation**, 11

This offers an axiomatic illustration of how the two-dimensional entry for OR is an isotope of the standard boolean \vee .

4 Deontic Modality

4.1 Two-Dimensional Deontic Logic

Here is an interim summary of where we are. In §§ 2.1–2.2, we presented several motivations for the two-dimensional framework as well as four notions of consequence that can be distinguished within it. The axiomatization of this framework was given in § 2.3. In § 2.4, we motivated exploring Åqvist logics with their (Å)-compliant atomics. Finally, in § 3, we sketched a two-dimensional semantics of disjunction and presented an axiomatization for it.

With this setup, we are ready to enrich the language with the deontic operators O and M . We do this by enriching the modal domain of our matrices with a deontic accessibility relation R_O . The goal of this section is to articulate the logic that results from lifting standard one-dimensional deontic logic into two-dimensions. We then show how doing so can solve the deontic puzzles with which we began.

First, however, we face a choice point regarding the deontic accessibility relation, similar to the choice point faced by the fully general vs. Åqvist approach to atomics sketched in § 2.4, viz., should the deontic accessibility relation be uniform across each row of the matrix? A ‘yes’ answer suggests that at its foundations, deontic accessibility cannot depend on facts which are (metaphysically or epistemically) indeterminate relative to the context set. A ‘no’ answer, on the other hand, suggests that it can.

A number of complications arise when one tries to adopt the latter non-uniform approach. For instance, recall that in § 2.2 we saw one motivation for going two-dimensional was to adopt a Stalnakerian theory of assertion that could accommodate sentences whose propositional content was not uniform. To do this, we require that assertions be diagonalized prior to update. But it is not clear how to “diagonalize” deontic claims if the accessibility relation is not uniform. What settles whether v is deontically accessible from w in a context set (a set of *worlds*) if whether v is deontically accessible from w varies with the world-as-actual, row-by-row? Thus, though we will present completeness results for both paths from this choice point, we will keep to the spirit of uniformity in that we will pursue only a uniform approach as route to explaining **Ross’s Puzzle** and **FCP**.

Second, there is another choice point concerning the properties of the deontic accessibility relation. In standard one-dimensional deontic logic, it is assumed that the deontic accessibility relation is at the very least a serial and shift-reflexive subrelation of R_{\square} .

May Implies Can. $\forall w, v: wR_O v \Rightarrow wR_{\square} v$

Seriality. $\forall w \exists v: wR_O v$

Shift-Reflexivity. $\forall w, v: wR_O v \Rightarrow vR_O v$

These constraints correspond respectively to the following axioms (where \hat{O} is defined to be the dual of O ; we separate \hat{O} from M conceptually for reasons that will be made clear in § 4.3):

$$(MC) \hat{O}p \rightarrow \diamond p$$

$$(D_O) Op \rightarrow \hat{O}p$$

$$(OT_O) O(Op \rightarrow p)$$

Adding (D_O) and (OT_O) to the minimal normal modal logic results in the modal logic KDU, i.e., “standard deontic logic+” in the terminology of McNamara 2010.

Stronger constraints could be imposed on R_O , though. In particular, it is natural to assume that R_O is **rigid**, meaning which worlds are “deontically ideal” does not vary from world to world:

$$\text{Deontic Rigidity. } \forall w, v, u: wR_{\square}v \ \& \ wR_Ou \Rightarrow vR_Ou$$

This corresponds to the following axiom:

$$(DR) \hat{O}p \rightarrow \square \hat{O}p$$

On this analysis, R_O effectively partitions W into the classes of deontically ideal and nonideal worlds.²³ This allows us to reduce deontic logic to alethic logic via the standard Anderson-Kanger reduction using \square and a privileged atomic d standing for “The deontic ideals are met” (Anderson, 1958; Kanger, 1971):

$$O\phi := \square(d \rightarrow \phi)$$

$$M\phi := \diamond(d \wedge \phi)$$

Again, for the sake of neutrality, we present completeness results for both options. It turns out that this option affects what we say about **FCP**. Because Fusco only adopts the weaker framework, without **Deontic Rigidity**, she needs to revise the relationship between O and M in order to explain **FCP**. Assuming **Deontic Rigidity**, however, this revision is not required.

Let us now turn to the question of how to axiomatize each combination of choices.

Definition 6. The two-dimensional deontic language \mathcal{L}^{D2D} is defined recursively as follows:

$$\phi ::= \text{Prop} \mid \neg\phi \mid (\phi \wedge \phi) \mid \square\phi \mid @\phi \mid \dagger\phi \mid O\phi.$$

In addition to the other abbreviations, we define $\hat{O}\phi := \neg O\neg\phi$.

²³This is similar to the way the deontic selection function of MacFarlane and Kolodny (2010) works. See also Kratzer 1977, 1981’s ordering-source approach.

Definition 7. A **deontic matrix frame** is a tuple $\mathcal{F} = \langle W \times W, R_{\square}, R_{@}, R_{\dagger}, R_O \rangle$, where $\langle W, R_{\square}, R_{@}, R_{\dagger} \rangle$ is a basic matrix frame and R_O is a serial, shift-reflexive subrelation of R_{\square} . A **deontic matrix model** is a pair of a deontic matrix frame with a valuation function $V: \text{Prop} \rightarrow \wp(W \times W)$. A deontic matrix frame (model) is **rigid** if in addition R_O satisfies **Deontic Rigidity**. Let DM be the class of deontic matrix models and DMR the class of rigid deontic matrix models.

Definition 8. Let **D2D** be the logic that results from adding **(MC)**–**(OT_O)** as axioms to **2D**, and let **D2DR** be the result of adding **(DR)** as well.

The following is proved in § A.

Theorem 3.

- (a) **D2D** is sound and complete for DM.
- (b) **D2DR** is sound and complete for DMR.

Now for adding the other options.

Definition 9. Let $\mathcal{M} = \langle W \times W, R_{\square}, R_{@}, R_{\dagger}, R_O, V \rangle$ be a deontic matrix model. Say R_O is **uniform** if the following condition is met for all $x, y, w, v \in W$:

$$\langle \langle x, w \rangle, \langle x, v \rangle \rangle \in R_O \quad \Leftrightarrow \quad \langle \langle y, w \rangle, \langle y, v \rangle \rangle \in R_O.$$

We'll say \mathcal{M} is **uniform** if R_O is. The definition of Åqvist matrix models from Definition 4 carries over to deontic matrix models. Let DMÅU be the class of Åqvist uniform deontic matrix models and DMÅUR the class of Åqvist, uniform, and rigid deontic matrix models.

To state the proof system, we first need a definition.

Definition 10. A \mathcal{L}^{D2D} -formula is **explicitly 1D** if it is @-free and †-free.

Definition 11. Let **F** := **D2DÅU** be the proof system axiomatized by Modus Ponens, Necessitation for each primitive modal operator, (**Å**), and the following axiom schemas:

- (D2D) ϕ , where ϕ is a theorem of **D2D**.
- (U) $O\phi \leftrightarrow \dagger O\phi$ where ϕ is an explicitly 1D-formula.

Let **FR** := **D2DÅUR** be defined similarly except we replace **(D2D)** with:

- (D2DR) ϕ , where ϕ is a theorem of **D2DR**.

F is the proof system that axiomatizes Fusco (2015)'s semantics, while **FR** is the same proof system extended with **(DR)**. The following is proved in § B.

Theorem 4.

- (a) **F** is sound and complete for $\text{DMA}\dot{\cup}$.
- (b) **FR** is sound and complete for $\text{DMA}\dot{\cup}\text{R}$.

F and **FR** give us a pair of one-dimensional deontic logics “lifted” into two dimensions—that is, into the space of matrices—in a way that preserves the original, one-dimensional interpretation of atoms.²⁴

4.2 Ross in the context of Deontic Matrix Models

Now that we’ve seen various ways of lifting deontic logic into two dimensions, what happens when we adopt Fusco’s two-dimensional semantics for **OR**? As it turns out, both **Ross’s Puzzle** and **FCP** can be accounted for (though, as we’ll see, the latter may require a bit extra work to obtain).

Start with **Ross’s Puzzle**. Even in **FR**, the Ross inference is not a priori valid: $Op \not\vdash_{\text{DMA}\dot{\cup}\text{R}} O(p \text{ OR } q)$. A counterexample is given in **Figure 6**. Let us say a model \mathcal{M} **a priori satisfies** ϕ , written $\mathcal{M} \triangleright \phi$, if $\mathcal{M}, z, z \Vdash \phi$ for all $z \in W$. (Thus, $\Gamma \triangleright_{\mathcal{C}} \phi$ iff for every $\mathcal{M} \in \mathcal{C}$, if $\mathcal{M} \triangleright \Gamma$, then $\mathcal{M} \triangleright \phi$.) The deontic matrix model in **Figure 6** a priori satisfies the premise Op . But it fails to a priori satisfy the conclusion $O(p \text{ OR } q)$. The witness for this failure is $\langle w_q, w_q \rangle$: at $\langle w_q, w_q \rangle$, the disjunction $p \text{ OR } q$ expresses the proposition that q . But q is not obligatory (or even permissible) at $\langle w_q, w_q \rangle$.

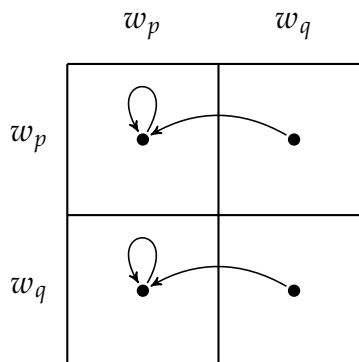


Figure 6: $\mathcal{M} \triangleright Op$

To get a better sense of why failures of the Ross inference arise, it helps to contrast this counterexample with a model where $O(p \text{ OR } q)$ is a priori satisfied,

²⁴These logics hence preserve the one-dimensional notion of “good arguments” from [Gregory 2001](#) and [Williamson 2013](#) (glossed in [footnote 20](#)).

such as in [Figure 7](#). We can see that q —the actually true answer to ‘ p , or q ?’ in w_q —is obligatory from the point of view of $\langle w_q, w_q \rangle$.²⁵ We can also see that p —the actually true answer to ‘ p , or q ?’ in w_p —is obligatory from the point of view of $\langle w_p, w_p \rangle$, the other diagonal, or ‘a priori’, point in the model. If it is q which the agent actually does—settling that she is at the diagonal point on the second row of the matrix—then it is q she ought to do. But the reverse is true if it is p which she actually does. Since the choice of actuality is up to her, what is deontically ideal is also up to her: she has a deontically *free choice* in [Figure 7](#) which she lacks in [Figure 6](#).²⁶

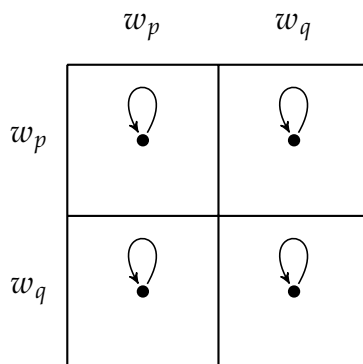


Figure 7: $\mathcal{M} \triangleright O(p \text{ OR } q)$

4.3 Permissibility and Free Choice Permission

Let’s turn now to **FCP**. We’ll start with deontically rigid models, since the explanation of **FCP** over this class is simpler. Three ingredients are required. First, M is the dual of O , i.e., $M := \hat{O}$. Second, the deontic accessibility relation is uniform. And third, the formulas involved in the inference are \hat{A} qvist, i.e., their propositional content does not vary with the world-as-actual. Thus, where the \hat{A} operator is defined as in [§ 2.4](#), we have the following free choice theorem:²⁷

Theorem 5. $M(\phi \text{ OR } \psi), \diamond(\phi \wedge \neg\psi), \hat{A}\phi, \hat{A}\psi \triangleright_{\text{DMAUR}} M\phi$.

²⁵To simplify the visuals, p and q are represented in [Figures 6–7](#) as mutually exclusive and jointly exhaustive.

²⁶It is worth noting that a proposal for understanding **FCP** which is similar in spirit but quite different in implementation to the present approach, is independently developed in the linguistics literature by [Kaufmann \(2016\)](#). Unfortunately, we lack the space to compare the similarities and differences of the views at length here.

²⁷In fact, [Theorem 5](#) holds even for the class of uniform and rigid deontic matrix models—that is, we don’t have to require valuations to be \hat{A} qvist. But the completeness proof of **F(R)** requires the deontic matrix models be \hat{A} qvist, so we’ve stated the theorem in terms of \hat{A} qvist models.

Proof. Let \mathcal{M} be a \mathring{A} qvist, uniform, and rigid deontic matrix model such that:

- (i) $\mathcal{M} \triangleright M(\phi \text{ OR } \psi)$
- (ii) $\mathcal{M} \triangleright \diamond(\phi \wedge \neg\psi)$
- (iii) $\mathcal{M} \triangleright \mathring{A}\phi \wedge \mathring{A}\psi$

Let $w \in W$; we will show that $\mathcal{M}, w, w \Vdash M\phi$. By (ii), there exists a $v \in W$ such that $\mathcal{M}, w, v \Vdash \phi \wedge \neg\psi$. By (iii), $\mathcal{M}, w, v \Vdash \phi \wedge \neg\psi$ iff $\mathcal{M}, w, v \Vdash \dagger\phi \wedge \neg\dagger\psi$, which holds iff $\mathcal{M}, v, v \Vdash \phi \wedge \neg\psi$. Hence, $\text{Ans}(v, \phi, \psi) = \{\phi\}$. By (i), $\mathcal{M}, v, v \Vdash M(\phi \text{ OR } \psi)$. So there exists a $u \in W$ such that $\langle v, v \rangle R_O \langle v, u \rangle$ and $\mathcal{M}, v, u \Vdash \phi \text{ OR } \psi$. Since $\text{Ans}(v, \phi, \psi) = \{\phi\}$, that means $\mathcal{M}, v, u \Vdash \phi$. By (iii), $\mathcal{M}, w, u \Vdash \phi$. But because R_O is uniform, $\langle w, v \rangle R_O \langle w, u \rangle$. And because R_O is rigid, that means $\langle w, w \rangle R_O \langle w, u \rangle$. Hence, $\mathcal{M}, w, w \Vdash M\phi$. \square

We can illustrate this using by contrasting [Figure 8](#) and [Figure 6](#). In [Figure 8](#), $M(p \text{ OR } q)$ is a priori satisfied because no matter which world is actual, the actual answer to ‘ p , or q ?’ is permissible. By contrast, in [Figure 6](#), q is not permissible, and so $M(p \text{ OR } q)$ is not a priori satisfied—specifically, it’s not satisfied at $\langle w_q, w_q \rangle$.

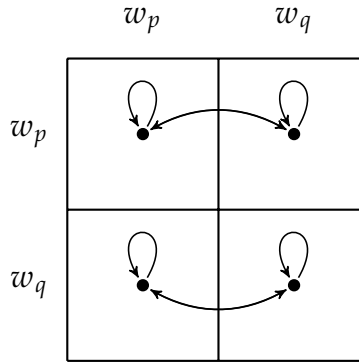


Figure 8: $\mathcal{M} \triangleright M(p \text{ OR } q)$

The addition of premises $\mathring{A}\phi$ and $\mathring{A}\psi$ is new relative to [Fusco 2015](#). There, Fusco states the free choice inference (using our notation) as:

$$M(\phi \text{ OR } \psi), \diamond(\phi \wedge \neg\psi), \diamond(\neg\phi \wedge \psi) \triangleright M\phi \wedge M\psi$$

where ϕ and ψ are both disjunction-free and non-modal. In that framework, Fusco implicitly assumes that all atomics are \mathring{A} qvist, which entails that any such formulas are \mathring{A} qvist. Thus, our theorem is a generalization of Fusco’s result: her free choice inference can be extended to *any* ϕ and ψ so long as their content does not vary from row-to-row.

There are cases involving non-Åqvist disjuncts, however, where free choice does not hold in this semantics. Here is a counterexample. Our matrix model \mathcal{M} will consist of four worlds $w_{pq}, w_{\bar{p}q}, w_{p\bar{q}}, w_{\bar{p}\bar{q}}$. Our (Åqvist) valuation will be the obvious one:

$$\begin{aligned} \langle w_{\alpha\beta}, w_{\gamma\delta} \rangle \in V(p) &\Leftrightarrow \gamma = p \\ \langle w_{\alpha\beta}, w_{\gamma\delta} \rangle \in V(q) &\Leftrightarrow \delta = q. \end{aligned}$$

The deontic accessibility relation will say all and only \bar{p} -worlds are deontically ideal:

$$\langle w_{\alpha\beta}, w_{\gamma\delta} \rangle R_O \langle w_{\alpha\beta}, w_{\gamma'\delta'} \rangle \Leftrightarrow \gamma' = \bar{p}$$

Note, R_O is both uniform and rigid. This model is summarized in [Figure 9](#).

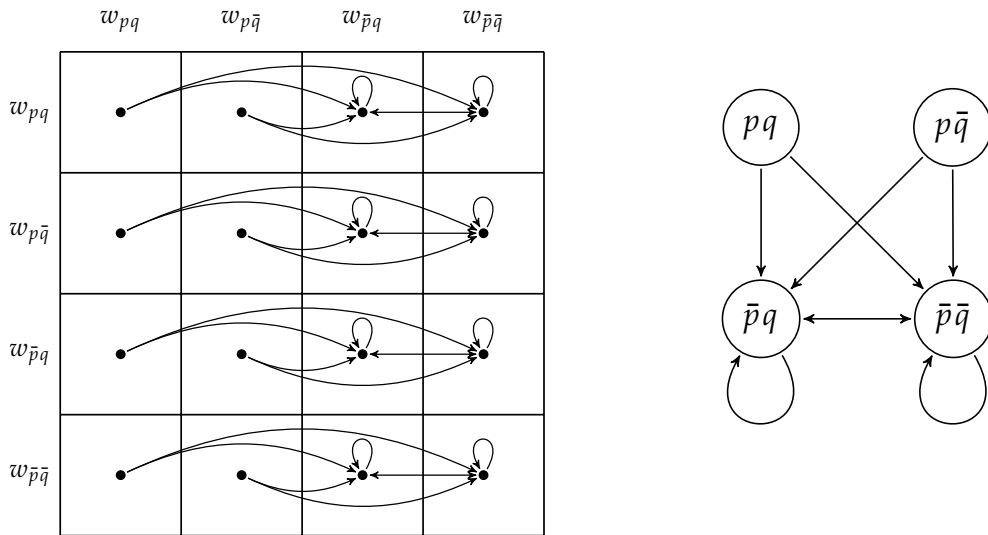


Figure 9: Counterexample to free choice with non-Åqvist disjuncts. The diagram on the right is a visualization of the accessibility relations on each row.

Then letting $\phi = @p \leftrightarrow p$ and $\psi = @q \leftrightarrow q$, we have:²⁸

$$\begin{aligned} \mathcal{M} &\triangleright M((@p \leftrightarrow p) \text{ OR } (@q \leftrightarrow q)) \\ \mathcal{M} &\triangleright \diamond((@p \leftrightarrow p) \wedge \neg(@q \leftrightarrow q)) \\ \mathcal{M} &\triangleright \diamond(\neg(@p \leftrightarrow p) \wedge (@q \leftrightarrow q)) \\ \mathcal{M} &\not\triangleright M(@p \leftrightarrow p) \end{aligned}$$

Thus, **FCP** does not hold for all non-Åqvist cases in Fusco's semantics.

With that said, we think this is not a serious cost to Fusco's approach, for two reasons. First, these sorts of counterexamples are quite strange. It is hard to see how we could have robust and reliable intuitions about their analogues in natural language. This is especially complicated by evidence that the actually operator @ is not be an adequate formalization of the English word 'actually'.²⁹ So these consequences of Fusco's semantics would likely be hard to test empirically.

Second, there is an alternative formulation of free choice available that avoids these complications. We can drop the premises $\mathring{A}\phi$ and $\mathring{A}\psi$ from the inference so long as we *diagonalize* the disjuncts first.

Theorem 6. $M(\dagger\phi \text{ OR } \dagger\psi), \diamond(\dagger\phi \wedge \neg\dagger\psi) \triangleright_{\text{DMAUR}} M\dagger\phi$

Proof. Immediate from Theorem 5 and the fact that $\models_{\text{DM}} \mathring{A}\dagger\phi$. □

In fact, Theorem 5 can be seen as a special instance of this more general principle, since $\mathring{A}\phi \models \square(\phi \leftrightarrow \dagger\phi)$. We often diagonalize material in the scope of modals as part of a broader reinterpretation strategy. For instance, while (9) is not metaphysically necessary, and thus (10) is false so-interpreted, there does seem to be a reading of (10) on which it is true, viz., one where the flavor of necessity is a priority.

- (9) I am here now.
- (10) Necessarily, I am here now.
 - a. False: $\square(\text{I am here now})$
 - b. True: $\square\dagger(\text{I am here now})$

It would therefore be unsurprising if diagonalization occurred in the scope of disjunction, given that disjunction is itself interpreted as a kind of binary modal.

²⁸To see why, note that for each $v \in W$, $\text{Ans}(v, @p \leftrightarrow p, @q \leftrightarrow q) = \{@p \leftrightarrow p, @q \leftrightarrow q\}$. This is because $@\phi \leftrightarrow \phi$ is a diagonal validity. Thus, for each $v \in W$, we have $\mathcal{M}, v, v \Vdash M((@p \leftrightarrow p) \text{ OR } (@q \leftrightarrow q))$ iff $\exists \delta \in \{q, \bar{q}\}: \mathcal{M}, v, w_{\bar{p}\delta} \Vdash (@p \leftrightarrow p) \vee (@q \leftrightarrow q)$. It is easily verified this holds for all v , since some deontically ideal world agrees with v on q . Moreover, the possibility premises are satisfied on every row, since every combination of truth values to p and q is realized. But while $\mathcal{M} \triangleright M(@q \leftrightarrow q)$, we do not have $M(@p \leftrightarrow p)$ satisfied at every diagonal point. Specifically, $\mathcal{M}, w_{pq}, w_{p\bar{q}} \not\Vdash M(@p \leftrightarrow p)$ and $\mathcal{M}, w_{p\bar{q}}, w_{p\bar{q}} \not\Vdash M(@p \leftrightarrow p)$.

²⁹For data showing that English 'actually' is more complex, see Yalcin (2015).

This concludes the explanation of free choice when the accessibility relation is deontically rigid. Let us now turn to the case where it is not. In fact, [Fusco \(2015\)](#) does not endorse deontic rigidity because of cases like the following:

Choosing Childbearing. You face a choice between conceiving a child early in your life [= p] or a different child significantly later [= q]. You believe that your values will be transformed by the choice you make. In particular, because the choice you make will be a necessary condition for the existence of a person you will love, you will affirm *that* choice over any other. There is no single psychological standpoint that values both of these potential persons to equal degree. ([Parfit, 1984](#), pg. 360–361; [Paul, 2014](#), Ch. 3; [Paul, 2015](#).)

Nice Choices at the Spa. Aromatherapy [= p] or body-wrap [= q]—which is it to be? You believe that, whichever you choose, you will be very glad you chose it. Mid-aromatherapy, the aromatherapy will seem self-evidently superior [to the body-wrap]. Mid-body-wrap, the body-wrap will seem self-evidently superior [to the aromatherapy]. ([Hare and Hedden, 2016](#), pg. 3)

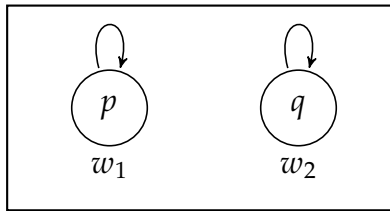


Figure 10: A one-dimensional nice choice

Suppose “nice choices” like the ones described above are possible. Then worlds where the agent makes different choices may have different deontic points of view. This would require abandoning **Deontic Rigidity** as a constraint on the deontic accessibility relation.

To accommodate free choice inferences in this more general framework, Fusco proposes to capture **FCP** by reconceiving of the connection between obligation and permission. In brief, where the standard view is that M is just \hat{O} , i.e., the dual of O , Fusco proposes to analyze M in terms of a weaker condition, viz., $\diamond\hat{O}$. Thus, the revised truth conditions for M are as follows:

(M_2) $y, x \Vdash M\phi$ iff there is some x' such that $x'R_Ox'$ and $y, x' \Vdash \phi$

Redefining M in this way, we obtain free choice once again:

Theorem 7. Where $M := \diamond\hat{O}$:

- (a) $M(\phi \text{ OR } \psi), \diamond(\phi \wedge \neg\psi), \mathring{A}\phi, \mathring{A}\psi \triangleright_{\text{DM}\hat{A}\text{U}} M\phi$
- (b) $M(\dagger\phi \text{ OR } \dagger\psi), \diamond(\dagger\phi \wedge \neg\dagger\psi) \triangleright_{\text{DM}\hat{A}\text{U}} M\dagger\phi$

Proof. The proof of (a) is the same as that of Theorem 5 up to where we inferred that $\mathcal{M}, v, v \Vdash M(\phi \text{ OR } \psi)$. From there, it follows that there exists a $u \in W$ such that for some $u' \in W$, $\langle v, u \rangle R_O \langle v, u' \rangle$ and $\mathcal{M}, v, u' \Vdash \phi \text{ OR } \psi$. Since $\text{Ans}(v, \phi, \psi) = \{\phi\}$, that means $\mathcal{M}, v, u' \Vdash \phi$. By (iii), $\mathcal{M}, w, u' \Vdash \phi$. But because R_O is uniform, $\langle w, u \rangle R_O \langle w, u' \rangle$. Hence, $\mathcal{M}, w, u \Vdash \hat{O}\phi$. So $\mathcal{M}, w, w \Vdash \diamond\hat{O}\phi = M\phi$. The proof of (b) is immediate as before. \square

Thus, one does not need to commit oneself to the claim that each world must agree with every other about which worlds are deontically ideal in order to capture free choice, so long as we take a revisionary stance on the relationship between ‘may’ and ‘ought’.

Note that, in deontically rigid models, this definition of M is *equivalent* to \hat{O} :

Theorem 8. $\models_{\text{DMR}} \hat{O}\phi \leftrightarrow \diamond\hat{O}\phi$.

Thus, one could in principle adopt this definition ($M := \diamond\hat{O}$) in **FR** as well; it’s just that doing so would not be a genuine departure from the orthodox view that ‘may’ is the dual of ‘ought’.

5 Conclusion

In this paper, we explored several ways of axiomatizing \square , \dagger , $@$, and the deontic operators O and M , with an eye to **FCP** and **Ross’s Puzzle**, following the path sketched by Fusco (2015, 2019). The language also provides a way of connecting the a priori (A) operator to Stalnaker’s “Assertion”, via the idea that a natural notion of global or informational consequence is in fact a diagonalized one as well. We axiomatized Fusco’s logic of disjunction using rules that collapse into the standard disjunction introduction/elimination rules in the one-dimensional setting. We then explored several different formulations of two-dimensional deontic logic and how these different choice points relate to the deontic puzzles in § 1.

While our focus has primarily been on deontic logic, the various choice points in § 4 highlight a broader lesson for the study of two-dimensional logics. When lifting a one-dimensional system into two dimensions, one needs to keep in perspective the original motivations for doing so. For some applications, it will be desirable not to impose uniformity on accessibility relations, thus preserving the full generality of the two-dimensional framework. But for other applications to natural language, especially in the context of a Stalnakerian picture of assertion, uniformity may be

motivated by more than mere convenience. We leave further discussion of these issues for another time.

A Axiomatizing D2D

In this appendix, we prove Theorem 3: **D2D(R)** is sound and complete for the class $\text{DM}(\mathbf{R})$ of (rigid) deontic matrix models. The proof is an extension of the completeness results in Fritz 2014; Fusco forthcoming for two-dimensional languages without deontic operators. For this reason, we will skip some of the proofs and refer the reader to those articles for details.

The proof strategy is in three steps. We identify three classes of frames:

1. $\text{Fr}_{\mathbf{D2D}(\mathbf{R})}$, the class of **D2D(R)**-frames;
2. $\mathbf{R}_{\mathbf{D}(\mathbf{R})}$, an intermediate class of (rigid) deontic Restall frames;
3. $\text{MF}_{\mathbf{D}(\mathbf{R})}$, the class of (rigid) deontic matrix frames.

The first step is to establish the soundness and completeness of **D2D(R)** over $\text{Fr}_{\mathbf{D2D}(\mathbf{R})}$. This follows immediately from Sahlqvist’s theorem, since all the axioms are Sahlqvist formulas.

Theorem 9. **D2D(R)** is sound and strongly complete with respect to $\text{Fr}_{\mathbf{D2D}(\mathbf{R})}$.

Proof. By Sahlqvist’s Theorem (Blackburn et al., 2002, Ch. 4). □

The second step is to show that $\text{Fr}_{\mathbf{D2D}(\mathbf{R})}$ is modally equivalent to an intermediate class of frames $\mathbf{R}_{\mathbf{D}(\mathbf{R})}$, viz., (rigid) deontic Restall frames. This is established by showing that $\mathbf{R}_{\mathbf{D}(\mathbf{R})}$ is the class of point-generated subframes of $\text{Fr}_{\mathbf{D2D}(\mathbf{R})}$. The final step is then to show that every deontic Restall frame is equivalent to a matrix frame by showing how to construct a bounded morphism from an arbitrary deontic Restall frame into a deontic matrix frame.

$$\text{Fr}_{\mathbf{D2D}(\mathbf{R})} \xrightarrow{\text{gen. submodel}} \mathbf{R}_{\mathbf{D}(\mathbf{R})} \xrightarrow{\text{bounded morphism}} \text{MF}_{\mathbf{D}(\mathbf{R})}$$

We start with some facts about **D2D**, which are all left as an exercise to the reader:

Fact 1. The following are all provable in **2D**:

$$\begin{array}{ll}
(X) \ \dagger(@\phi \leftrightarrow \phi) & (\text{Red}_\dagger) \ \dagger\dagger\phi \leftrightarrow \dagger\phi \\
(Y) \ @(\dagger\phi \leftrightarrow \phi) & (\text{Red}_@) \ @@\phi \leftrightarrow @\phi \\
(\dagger\diamond\dagger) \ \dagger\diamond\dagger\phi \leftrightarrow \diamond\dagger\phi & (A@) \ A\phi \rightarrow @\phi
\end{array}$$

In addition, the following is provable in **D2D**:

$$(@5_\diamond) \ \hat{O}@\phi \rightarrow @\phi$$

Definition 12. A **Kripke frame** is a tuple $\mathcal{F} = \langle W, R_\square, R_@, R_\dagger, R_O \rangle$ where W is a set of points and each R_Δ is a binary relation on W .

Fact 2. The axioms of **D2D(R)** have the following (global) first-order correspondents in Kripke frames:

$$\begin{array}{ll}
(T_\square) \ R_\square \text{ is reflexive} & (Y_\rightarrow) \ \forall w, v, u: \\
& \quad (wR_@v \ \& \ vR_\dagger u) \Rightarrow v = u \\
(5_\square) \ R_\square \text{ is Euclidean} & (4_A) \ R_\square \circ R_\dagger \text{ is transitive} \\
(@5_\diamond) \ \forall w, v, u: & (5_A) \ \forall x, y, a, b: \\
& \quad (wR_\square v \ \& \ wR_@u) \Rightarrow vR_@u & \quad (xR_\square y \ \& \ xR_\square a \ \& \ aR_\dagger b) \Rightarrow \\
& & \quad \exists c(bR_\square c \ \& \ \forall d(cR_\dagger d \Rightarrow yR_\dagger d)) \\
(G) \ R_@ \subseteq R_\square & (Z) \ R_O \subseteq R_\square \\
(R_@) \ R_@ \text{ is a function} & (D_O) \ R_O \text{ is serial} \\
(R_\dagger) \ R_\dagger \text{ is a function} & (U_O) \ R_O \text{ is shift-reflexive} \\
(X_\rightarrow) \ \forall w, v, u: & (DR) \ \forall w, v, u: \\
& \quad (wR_\dagger v \ \& \ vR_@u) \Rightarrow v = u & \quad (wR_\square v \ \& \ wR_O u) \Rightarrow vR_O u
\end{array}$$

The correspondents for (X) and (Y) in Fact 1 are as follows:

(X) Within $Img(R_\dagger)$, $R_@$ is the identity relation

(Y) Within $Img(R_@)$, R_\dagger is the identity relation

The following facts are all proven in [Fusco forthcoming](#):

Fact 3. $Img(R_@) = Img(R_\dagger)$.

Fact 4. $\forall d_1, d_2 \in Img(R_@)$: if $d_1(R_\square \circ R_\dagger)d_2$, then $d_2(R_\square \circ R_\dagger)d_1$.

Fact 5. $R_\square \circ R_\dagger$ an equivalence relation on $Img(R_@)$.

Fact 6. Suppose $wR_@d_1$ and $wR_\dagger d_2$. Then $\exists w'$ s.t. $w'R_@d_2$ and $w'R_\dagger d_1$.

Before we show $\text{Fr}_{\text{D2D}(\mathbf{R})}$ is modally equivalent to $\text{R}_{\text{D}(\mathbf{R})}$, it will help to prove the following lemma. (Notation: if R_Δ is a function, we use " $R_\Delta(x)$ " for the unique y such that $xR_\Delta y$)

Lemma 1. For any point-generated subframe $\mathcal{F}_w = \langle W', R'_\square, R'_\dagger, R'_\@, R'_\circ \rangle \in \text{Fr}_{\text{D2D}}$: $W' = R_\square[R_\dagger[R_\square[\{w\}]]]$.

Proof. We let $X := R_\square[R_\dagger[R_\square[\{w\}]]]$. We show (1) $w \in X$ and (2) that for each operator ∇ that $R_\nabla[X] \subseteq X$.

(1) Let $u = R_\@(w)$. By (G), $wR_\square u$. By (Y), $uR_\dagger u$. By symmetry of R_\square , $uR_\square w$. Hence via u , it follows that $w \in X$.

(2- \square) Since R_\square is transitive, $R_\square[X] = X$.

(2- \circ) Since $R_\circ \subseteq R_\square$, it follows that $R_\circ[X] \subseteq R_\square[X] \subseteq X$.

(2- \dagger) By (4_A), $R_\dagger[X] = R_\dagger[R_\square[R_\dagger[R_\square[\{w\}]]]] = R_\dagger[R_\square[\{w\}]]$. By (T \square), $R_\dagger[R_\square[\{w\}]] \subseteq R_\square[R_\dagger[R_\square[\{w\}]]] = X$.

(2- $\@$) By (A $\@$), $R_\@[X] \subseteq R_\dagger[R_\square[X]] \subseteq R_\dagger[R_\square[R_\square[R_\dagger[R_\square[\{w\}]]]] = R_\dagger[R_\square[R_\dagger[R_\square[\{w\}]]]]$. By (4_A) and (T \square), $R_\dagger[R_\square[R_\dagger[R_\square[\{w\}]]]] \subseteq R_\dagger[R_\square[\{w\}]] \subseteq R_\square[R_\dagger[R_\square[\{w\}]]] = X$.

□

Definition 13. A (rigid) deontic Restall Frame is a frame $\mathcal{R} = \langle W, R_\square, R_\@, R_\dagger, R_\circ \rangle$ such that

1. R_\square is an equivalence relation
2. $R_\@$ is a function such that
 - (a) $wR_\@v \rightarrow wR_\square v$
 - (b) $R_\@$ maps any two R_\square -related worlds to the same point
3. R_\dagger is a function such that:
 - (a) for any w : $R_\dagger[R_\square[\{w\}]] = \text{Im}g(R_\@)$
 - (b) R_\dagger is reflexive over $\text{Im}g(R_\@)$
4. R_\circ is a (rigid,) serial, shift-reflexive subrelation of R_\square .

Let $\text{R}_{\text{D}(\mathbf{R})}$ be the class of (rigid) deontic Restall frames.

Lemma 2. $\text{R}_{\text{D}(\mathbf{R})} \subset \text{Fr}_{\text{D2D}(\mathbf{R})}$.

Proof. To verify this, it suffices to go through the first-order correspondents of $\mathbf{D2D(R)}$ from Fact 2. \square

Lemma 3. Every point-generated subframe $\mathcal{F}_w \in \text{Fr}_{\mathbf{D2D(R)}}$ is a (rigid) deontic Restall frame.

Proof. Where $\mathcal{F}_w = \langle W', R'_{@}, R'_{\dagger}, R'_{\square}, R'_{\circ} \rangle$ is a w -generated subframe in $\text{Fr}_{\mathbf{D2D}}$, let v be an arbitrarily chosen world in W' . We want to establish that \mathcal{F}_w satisfies conditions 1–4 of deontic Restall frames. Conditions 1–3 are established in [Fusco forthcoming](#). Condition 4 follows from the correspondences in Fact 2. \square

Theorem 10. $\text{Fr}_{\mathbf{D2D(R)}}$ and $\mathbf{R}_{\mathbf{D(R)}}$ are modally equivalent.

Proof. By Lemma 2, any formula falsifiable in $\mathcal{R} \in \mathbf{R}_{\mathbf{D(R)}}$ is falsifiable in some $\mathcal{F} \in \text{Fr}_{\mathbf{D2D(R)}}$. By Lemma 3, any formula falsifiable in a point-generated subframe $\mathcal{F}_w \in \text{Fr}_{\mathbf{D2D(R)}}$ is falsifiable in some $\mathcal{R} \in \mathbf{R}_{\mathbf{D(R)}}$. But any formula falsifiable in a $\mathcal{F} \in \text{Fr}_{\mathbf{D2D(R)}}$ is falsifiable in a point-generated subframe ([Blackburn et al., 2002](#), Proposition 2.6.). \square

Now we just need to reduce $\mathbf{R}_{\mathbf{D(R)}}$ to $\mathbf{MF}_{\mathbf{D(R)}}$. Some notation:

- We write $R[Y]$ for the image of a set Y under R .
- Let $D = \text{Img}(R_{@})$. We will use $i, j \dots$ as indices over D .
- Where \mathcal{R} is a deontic Restall frame, let $C = \{c_1, \dots, c_n\}$ be the set of R_{\square} -cells in \mathcal{R} . Since there is one $d \in D$ in each such cell, C is the cardinality of D .

Fact 7. $\bigcup_{i \in D} c_i = W$.

Fact 8. There is a unique $R_{@}$ -fixed point in each R_{\square} -cell $c_i \subset W$.

Notation: call this $R_{@}$ -fixed point $R_{@}(c_i)$.

Fact 9. For any w and any $c_j \subseteq W$, $\exists! v \in c_j$ s.t. $wR_{\dagger}v$.

More notation: where $c_i \in C$, let c_i^j be the set $\{w \in c_i \mid \exists v \in c_j: wR_{\dagger}v\}$. Let $R_{\dagger}(c_i^j)$ be the unique $v \in c_j$ such that $\forall w \in c_i^j, wR_{\dagger}v$. That $R_{\dagger}(c_i^j)$ is a fixed point of R_{\dagger} follows from ([Red \$_{\dagger}\$](#)).

Corollary 1. Each c_i can be partitioned into $\{c_i^j \mid j \in D\}$, where c_i^j is a nonempty subset of c_i such that $\forall w \in c_i^j, wR_{\dagger}(R_{@}(c_j))$. Hence $\bigcup_{j \in D} c_i^j = c_i$, and $\bigcup_{i \in D} (\bigcup_{j \in D} c_i^j) = W$.

Theorem 11. $\mathbf{R}_{\mathbf{D(R)}}$ and $\mathbf{MF}_{\mathbf{D(R)}}$ are modally equivalent.

Proof. Since $\text{MF}_{D(\mathcal{R})} \subset \text{R}_{D(\mathcal{R})}$, every (rigid) deontic matrix frame is equivalent to some (rigid) deontic Restall frame. For the converse, given a deontic Restall frame \mathcal{R} with domain W and $\text{Img}(R_{@}) = D$, we will build a matrix frame \mathcal{M} and construct, row-by-row, a surjective bounded morphism f from \mathcal{M} to \mathcal{R} , from which modal equivalence follows (Blackburn et al., 2002, Proposition 2.14).

The points of our matrix frame \mathcal{M} will be $(D \times W) \times (D \times W)$. For any $i, j \in D$ and $y \in W$, let $A_{i,y}^j$ be the set $\{\langle i, y \rangle\} \times \{\langle j, x \rangle : x \in W\}$. Note that $|A_{i,y}^j| = |W|$.

For each $i, j \in D$ and $y \in W$, fix some surjective $g_{i,y}^j : W \rightarrow c_i^j$ such that if $i = j$ and $y = x$, then $g_{i,y}^j(x) = R_{@}(c_i)$. (This condition maps the “diagonal” points of the matrix frame into D in the deontic Restall frame.) We know such surjective functions exist since $|A_{i,y}^j| \geq |c_i^j|$.

The accessibility relations $R_{\square}^{\mathcal{M}}$, $R_{@}^{\mathcal{M}}$, and $R_{\dagger}^{\mathcal{M}}$ are already determined by the definition of a deontic matrix frame. To define $R_{\circ}^{\mathcal{M}}$, first define a function $F : (D \times W) \times (D \times W) \rightarrow W$ as follows: $F(\langle\langle i, y \rangle, \langle j, x \rangle\rangle) = g_{i,y}^j(x)$. We define $R_{\circ}^{\mathcal{M}}$ in terms of F :

$$\langle\langle i, y \rangle, \langle j, x \rangle\rangle R_{\circ}^{\mathcal{M}} \langle\langle i, y \rangle, \langle j', x' \rangle\rangle \Leftrightarrow F(\langle\langle i, y \rangle, \langle j, x \rangle\rangle) R_{\circ} F(\langle\langle i, y \rangle, \langle j', x' \rangle\rangle).$$

We will now show that F is a bounded morphism from \mathcal{M} to \mathcal{R} .³⁰ To do this, it suffices to go through the Back and Forth conditions for $\Delta \in \{\square, @, \dagger, O\}$ for each point. The only new case not in the proof from Fusco forthcoming is the O case.

(O , Forth) Immediate by the definition of $R_{\circ}^{\mathcal{M}}$.

(O , Back) Suppose $F(\langle\langle i, y \rangle, \langle j, x \rangle\rangle) R_{\circ} v'$. We want to show that there is some $\langle\langle i, y \rangle, \langle j', x' \rangle\rangle$ such that $\langle\langle i, y \rangle, \langle j, x \rangle\rangle R_{\circ}^{\mathcal{M}} \langle\langle i, y \rangle, \langle j', x' \rangle\rangle$ and also that $F(\langle\langle i, y \rangle, \langle j', x' \rangle\rangle) = v'$. To witness this existential, we can choose any $\langle\langle i, y \rangle, \langle j', x' \rangle\rangle$ such that $F(\langle\langle i, y \rangle, \langle j', x' \rangle\rangle) = v'$ (we know one exists since F is surjective).

Hence, F is a bounded morphism, and so \mathcal{M} is modally equivalent to \mathcal{R} . \square

³⁰ F is a bounded morphism if it satisfies the following conditions for each $\Delta \in \{\square, @, \dagger, O\}$ (Blackburn et al., 2002, pg. 59):

1. w and $F(w)$ satisfy the same proposition letters;
2. if $w R_{\Delta}^{\mathcal{M}} v$ then $F(w) R_{\Delta} F(v)$ (the *Forth condition*);
3. if $F(w) R_{\Delta} v'$ then there exists some v such that $w R_{\Delta}^{\mathcal{M}} v$ and $F(v) = v'$ (the *Back condition*).

B Axiomatizing Deontic Åqvist Logic

In this appendix, we prove Theorem 4: **F** is sound and complete for the class of Åqvist uniform deontic matrix models $\text{DM}\mathring{\text{A}}\text{U}$. (The proof that **FR** is sound and complete for $\text{DM}\mathring{\text{A}}\text{UR}$ is analogous.) Soundness is straightforward and left to the reader. For completeness, the strategy will be to bootstrap off the completeness result for **D2D** in § A.

First, we need to prove some lemmas about the expressive power of **D2D**.

Lemma 4. For any \mathcal{L}^{D2D} -formula ϕ , the following formulas are strictly valid over DM :

- (a) $\Box(@\phi \vee \psi) \leftrightarrow (@\phi \vee \Box\psi)$
- (b) $O(@\phi \vee \psi) \leftrightarrow (@\phi \vee O\psi)$.

Recall the definition of explicitly 1D formulas (Definition 10).

Lemma 5. For any deontic matrix model $\mathcal{M} = \langle W \times W, R_\Box, R_\@, R_\dagger, R_O, V \rangle$, if \mathcal{M} satisfies $(\mathring{\text{A}})$ and (U) everywhere, then for any $w, w', v \in W$, any explicitly 1D \mathcal{L}^{D2D} -formula ϕ :

$$\mathcal{M}, w, v \Vdash \phi \quad \Leftrightarrow \quad \mathcal{M}, w', v \Vdash \phi.$$

Proof. By induction. □

Definition 14. An **@-atom** is a \mathcal{L}^{D2D} -formula that is either explicitly 1D, or is of the form $@\phi$ where ϕ is explicitly 1D.

The following is an extension of a result for languages without \dagger or O proven in Hazen et al. 2013:

Lemma 6. Every \mathcal{L}^{D2D} -formula is strictly equivalent to a boolean combination of @-atoms over $\text{DM}\mathring{\text{A}}\text{U}$.

Proof. By induction on the structure of \mathcal{L}^{D2D} -formulas. Throughout, we'll use " $\phi \equiv \psi$ " to mean ϕ and ψ are strictly equivalent over $\text{DM}\mathring{\text{A}}\text{U}$. The atomic and boolean cases are trivial. We'll present the other cases. Assume for inductive hypothesis that the claim holds of ϕ . In particular, assume that:

$$\phi \equiv \bigwedge_{i=1}^k \bigvee_{j=1}^{n_i} \alpha_{i,j}$$

where each $\alpha_{i,j}$ is an @-atom.

(□) Since \square commutes with conjunction:

$$\square\phi \equiv \square \bigwedge_{i=1}^k \bigvee_{j=1}^{n_i} \alpha_{i,j} \equiv \bigwedge_{i=1}^k \square \bigvee_{j=1}^{n_i} \alpha_{i,j}$$

So it suffices to show that each $\square \bigvee_{j=1}^{n_i} \alpha_{i,j}$ is equivalent to a boolean combination of @-atoms. First, write $\bigvee_{j=1}^{n_i} \alpha_{i,j}$ as:

$$@\beta_{i,1} \vee \cdots \vee @\beta_{i,m} \vee \gamma_{i,m+1} \vee \cdots \vee \gamma_{i,j}$$

where each $\beta_{i,x}$ and $\gamma_{i,y}$ are explicitly 1D. Then by Lemma 4(a):

$$\begin{aligned} \square \bigvee_{j=1}^{n_i} \alpha_{i,j} &\equiv \square (@\beta_{i,1} \vee \cdots \vee @\beta_{i,m} \vee \gamma_{i,m+1} \vee \cdots \vee \gamma_{i,j}) \\ &\equiv @\beta_{i,1} \vee \cdots \vee @\beta_{i,m} \vee \square(\gamma_{i,m+1} \vee \cdots \vee \gamma_{i,j}) \end{aligned}$$

But now all these terms are @-atoms.

(O) Similar to the (□) case.

(@) Since @ commutes with booleans:

$$@\phi \equiv @ \bigwedge_{i=1}^k \bigvee_{j=1}^{n_i} \alpha_{i,j} \equiv \bigwedge_{i=1}^k \bigvee_{j=1}^{n_i} @\alpha_{i,j}$$

Now, either $\alpha_{i,j}$ is explicitly 1D, in which case $@\alpha_{i,j}$ is an @-atom, or $\alpha_{i,j} = @\beta_{i,j}$ where $\beta_{i,j}$ is explicitly 1D, in which case $@\alpha_{i,j} \equiv @@\beta_{i,j} \equiv @\beta_{i,j} \equiv \alpha_{i,j}$, so we can replace $@\alpha_{i,j}$ with $\alpha_{i,j}$. The result is therefore a boolean combination of @-atoms.

(†) Since † commutes with booleans:

$$\dagger\phi \equiv \dagger \bigwedge_{i=1}^k \bigvee_{j=1}^{n_i} \alpha_{i,j} \equiv \bigwedge_{i=1}^k \bigvee_{j=1}^{n_i} \dagger\alpha_{i,j}$$

Now, either $\alpha_{i,j}$ is explicitly 1D or $\alpha_{i,j} = @\beta_{i,j}$ where $\beta_{i,j}$ is explicitly 1D. In the former case, $\dagger\alpha_{i,j} \equiv \alpha_{i,j}$ by Lemma 5. In the latter case, $\dagger\alpha_{i,j} \equiv \dagger@\beta_{i,j} \equiv \dagger\beta_{i,j}$, which is equivalent to $\beta_{i,j}$ by Lemma 5 again. So either way, $\dagger\alpha_{i,j} \equiv \alpha_{i,j}$, which means ϕ is strictly equivalent to $\dagger\phi$ already.

□

Theorem 12. F is sound and complete for DMÅU.

Proof. Let Γ be a **F**-consistent set of formulas. Take a maximal **F**-consistent extension $\Gamma^+ \supseteq \Gamma$ (the proof that one exists is standard). By Theorem 3, there is a deontic matrix model \mathcal{M} and some $y, x \in W$ such that $\mathcal{M}, y, x \Vdash \Gamma^+$. We first show that the valuation V is already $\mathring{\text{Aqvist}}$. Then we show how to transform \mathcal{M} into an equivalent model whose deontic accessibility relation is uniform.

First, V is $\mathring{\text{Aqvist}}$: since $\Box\uparrow\Box(p \leftrightarrow \uparrow p) \in \Gamma^+$ for all $p \in \text{Prop}$, it follows that $\mathcal{M}, w, v \Vdash p \leftrightarrow \uparrow p$ for all $w, v \in X$. Henec, for all $w, w', v \in X$:

$$\begin{aligned} \langle w, v \rangle \in V(p) &\Leftrightarrow \mathcal{M}, w, v \Vdash p \\ &\Leftrightarrow \mathcal{M}, w, v \Vdash \uparrow p \\ &\Leftrightarrow \mathcal{M}, v, v \Vdash p \\ &\Leftrightarrow \mathcal{M}, w', v \Vdash \uparrow p \\ &\Leftrightarrow \mathcal{M}, w', v \Vdash p \\ &\Leftrightarrow \langle w', v \rangle \in V(p). \end{aligned}$$

Next, we show how to transform \mathcal{M} into an equivalent one whose deontic accessibility relation is uniform. Define R_O^\cup as follows:

$$\langle w, v \rangle R_O^\cup \langle w, u \rangle \Leftrightarrow \exists z: \langle z, v \rangle R_O \langle z, u \rangle$$

Define $\mathcal{M}^\cup = \langle W \times W, R_\Box, R_\otimes, R_\uparrow, R_O^\cup, V \rangle$. Clearly, R_O^\cup is uniform. (Note also that if R_O is rigid, so is R_O^\cup .)

Lemma 7. For all $w, v \in W$ and all explicitly 1D \mathcal{L}^{D2D} -formulas ϕ :

$$\mathcal{M}, w, v \Vdash \phi \Leftrightarrow \mathcal{M}^\cup, w, v \Vdash \phi.$$

Proof. By induction on the complexity of ϕ . We only present the O case (the others are straightforward). Suppose for inductive hypothesis that the claim holds of ϕ . Clearly, if $\mathcal{M}^\cup, w, v \Vdash O\phi$, then $\mathcal{M}, w, v \Vdash O\phi$ since $R_O \subseteq R_O^\cup$. So we just need to establish the converse.

Suppose $\mathcal{M}^\cup, w, v \not\Vdash O\phi$. Thus, for some $u \in W$, we have $\langle w, v \rangle R_O^\cup \langle w, u \rangle$ and $\mathcal{M}^\cup, w, u \not\Vdash \phi$. By definition of R_O^\cup , for some $z \in W$, $\langle z, v \rangle R_O \langle z, u \rangle$. And by inductive hypothesis, $\mathcal{M}, w, u \not\Vdash \phi$. Hence, by Lemma 5, $\mathcal{M}, z, u \not\Vdash \phi$. So $\mathcal{M}, z, v \not\Vdash O\phi$. But again by Lemma 5, $\mathcal{M}, w, v \not\Vdash O\phi$. \square

Lemma 8. For all $w, v \in W$ and all \otimes -atoms ϕ :

$$\mathcal{M}, w, v \Vdash \phi \Leftrightarrow \mathcal{M}^\cup, w, v \Vdash \phi.$$

Proof. If ϕ is an explicitly 1D \mathcal{L}^{D2D} -formula, then this is ensured by Lemma 7. If $\phi = \otimes\psi$ where ψ is explicitly 1D, then:

$$\begin{aligned} \mathcal{M}, w, v \Vdash \otimes\psi &\Leftrightarrow \mathcal{M}, w, w \Vdash \psi \\ &\Leftrightarrow \mathcal{M}^\cup, w, w \Vdash \psi \text{ (Lemma 7)} \\ &\Leftrightarrow \mathcal{M}^\cup, w, v \Vdash \otimes\psi. \end{aligned}$$

□

Hence, by Lemma 6, for any \mathcal{L}^{D2D} -formula ϕ and any $w, v \in W$: $\mathcal{M}, w, v \Vdash \phi$ iff $\mathcal{M}^\cup, w, v \Vdash \phi$. So $\mathcal{M}^\cup, y, x \Vdash \Gamma^+$. □

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