Foreign Exchange, ADR's and Quanto-Securities

These notes consider foreign exchange markets and the pricing of derivative securities in these markets. As is the case with equity derivatives, the implied volatility surface corresponding to vanilla European FX option prices is neither flat nor constant. It is therefore widely accepted that the Black-Scholes GBM model is a poor model for FX markets. As is the case with equity derivatives, however, vanilla FX option prices are quoted and their Greeks are calculated using the Black-Scholes framework. It is therefore necessary to understand how Black-Scholes applies to the FX markets when working with derivatives in these markets.

We will also consider assets that are denominated in a foreign currency but whose value we wish to determine in units of a domestic currency. There are two different means of converting the asset's foreign currency value into domestic currency: (i) using the prevailing exchange rate and (ii) using a fixed predetermined exchange rate. The latter method leads to the concept of a *quanto* security. We will study both methods in some detail. Because such securities are exotic, they are generally not priced in practice using the Black-Scholes framework. Nonetheless, we will consider them in this framework for two reasons: (i) given our knowledge of Black-Scholes, this is the easiest way of introducing the concepts and (ii) it will afford us additional opportunities to work with martingale pricing using different numeraires and EMMs.

A particular feature of FX markets are the triangular relationships that exist between currency triples. For example, the USD/JPY exchange rate can be expressed in terms of the USD/EUR and EUR/JPY exchange rates. These implications ought to be borne in mind when working with FX derivatives. They and other peculiarities of FX modeling will be studied further in the assignments.

1 Foreign Exchange Modeling

Unless otherwise stated we will call one currency the **domestic** currency and the other the **foreign** currency. We will then let X_t denote the exchange rate at time t, representing the time t cost in units of the domestic currency of 1 unit of the foreign currency. For example, taking the usual market convention, if the USD/EUR[1] exchange rate is 1.2, then USD is the domestic currency, EUR is the foreign currency and 1 EUR costs $\$1.20$. Suppose instead we expressed the exchange rate as the cost in Euro of $\$1$ USD. Then USD would be the foreign currency and EUR would be the domestic currency. Note that the designations "domestic" and "foreign" have nothing to do with where you are living or where the transaction takes place. They are a function only of how the exchange rate is quoted.

The terms **base** and **quote** are also often used in practice. The exchange rate then represents how much of the quote currency is needed to purchase one unit of the base currency. So[2] "quote = domestic" and "base = foreign". We will continue to use "domestic" and "foreign" in these lecture notes. Later on we will use "base currency" to refer to the accounting currency. For example, a US based company would take the USD as its base currency whereas a German company would have the EUR as its base currency.

We will let r_d and r_f denote the domestic and foreign risk-free rates[3], respectively. Note that the foreign currency plays the role of the "stock" and r_f is the "dividend yield" of the stock. Note also that holding the

[1] In these notes when we write **"Curr1/Curr2"** we will always take Curr1 to be the domestic currency and Curr2 to be the foreign currency. Unfortunately, in practice this is not necessarily the case and the domestic and foreign designations will depend on the currency pair convention.

[2] But beware: I have seen "base" used in place of the domestic currency as well!

[3] We will assume that r_d and r_f are constant in these notes. When pricing long-dated FX derivatives, these rates are often taken to be stochastic. The same applies for long-dated equity derivatives.

foreign cash account is only risk-free from the perspective of someone who uses the foreign currency as their *unit of account*, i.e. accounting currency. A domestic investor who invests in the foreign cash account, or any other asset denominated in the foreign currency, is exposed to exchange rate risk.

Currency Forwards

Let $F_t^{(T)}$ denote the time t price of a forward contract for delivery of the foreign currency at time T. Then since the initial value of the forward contract is 0, martingale pricing implies

$$0 \;=\; \mathbf{E}_t^{\mathcal{Q}}\left[\frac{F_t^{(T)} - X_T}{B_T}\right]$$

which then implies

$$F_t^{(T)} \;=\; \frac{\mathbf{E}_t^{\mathcal{Q}}\left[\frac{X_T}{B_T}\right]}{\mathbf{E}_t^{\mathcal{Q}}\left[\frac{1}{B_T}\right]}.$$

Assuming that interest rates are constant[4] we obtain

$$\begin{aligned} F_t^{(T)} &= \mathbf{E}_t^{\mathcal{Q}}\left[X_T\right] \\ &= e^{r_d(T-t)}\, \mathbf{E}_t^{\mathcal{Q}}\left[e^{-r_d(T-t)}\, X_T\right] \\ &= e^{(r_d - r_f)(T-t)} X_t \end{aligned} \tag{1}$$

a relationship that is sometimes referred to as *covered interest parity*. Note that (1) may also be derived directly using a standard no-arbitrage argument. The forward currency markets[5] are very liquid and play an important role in currency trading. A forward FX rate, $F_t^{(T)}$, is usually quoted as a *premium* or *discount* to the spot rate, X_t, via the *forward points*.

FX Swaps

An FX swap is a simultaneous purchase and sale of identical amounts of one currency for another with two different *value* dates. The value dates are the dates upon which delivery of the currencies take place. In an FX swap, the first date is usually[6] the *spot* date and the second date is some forward date, T. An FX swap is then a regular spot FX trade combined with a forward trade, both of which are executed simultaneously for the same quantity. FX swaps are often called **forex** swaps. They should not be confused with currency swaps which are considerably less liquid. A currency swap is typically a long-dated instrument where interest payments and principal in one currency are exchanged for interest payments and principal in another currency. FX swaps are regularly used by institutions to fund their FX balances.

Forwards and FX swaps are typically quoted in terms of **forward points** which are the difference between the forward price and the spot price. Using (1) we see

$$F_t^{(T)} - X_t \;=\; \left(e^{(r_d - r_f)(T-t)} - 1\right) X_t$$

When interest rates are identical the forward points are zero. As the interest rate differential gets larger, the absolute value of the forward points increases.

Currency Options

A European call option on the exchange rate, X, with maturity T and strike K pays $\max(0,\; X_T - K)$ at time T. If we assume that X_t has GBM dynamics then the same Black-Scholes analysis that we applied to options on

[4]We could also allow interest rates that are (i) deterministic or (ii) stochastic but $\mathcal{Q}$-independent of X_T in our derivation of (1). But a no-arbitrage argument using zero-coupon bonds could still be used to derive (1) in general. See Exercise 1.

[5]Exchange-traded currency futures also exist.

[6]But other value dates are also possible. It is possible, for example, to trade forward-forward swaps where both value dates are beyond the spot date, or *tom-next* swaps.

stocks will also apply here. In particular the $\mathcal{Q}$-dynamics of the exchange rate are given by

$$dX_t = (r_d - r_f)X_t \, dt + \sigma_x X_t \, dW_t^{(x)} \tag{2}$$

and the price of the currency option satisfies

$$C(X,t) = e^{-r_f(T-t)} X_t \Phi(d_1) - e^{-r_d(T-t)} K \Phi(d_2) \tag{3}$$

$$\text{where } d_1 = \frac{\log\left(\frac{X_t}{K}\right) + (r_d - r_f + \sigma_x^2/2)(T-t)}{\sigma_x \sqrt{T-t}}$$

$$\text{and } d_2 = d_1 - \sigma\sqrt{T-t}.$$

where σ_x is the exchange rate volatility. All the usual Black-Scholes Greeks apply. It is worth bearing in mind, however, that foreign exchange markets typically assume a *sticky-by-delta* implied volatility surface. This means that as the exchange rate moves, the volatility of an option with a given strike is also assumed to move in such a way that the volatility skew, as a function of delta, does **not** move.

The **notional** of the option is the number of *foreign* currency units that the option holder has the right to buy or sell at maturity. So for example, if the notional of a call option is N, then the time t value of the option is $N \times C(X,t)$.

Delta and the Option Premium

When trading a currency option, the price of the option may be paid in units of the domestic currency or it may be paid in units of the foreign currency. This situation never arises when trading stock options. For example, if you purchase an option on IBM then you will pay for that option in dollars, i.e. the domestic currency, and not in IBM stock which plays the role of the foreign currency. Note that this is a matter of practice as there is nothing in theory to stop me paying for the IBM option in IBM stock. In currency markets, however, and depending on the currency pair, it might be quite natural to pay for the currency option in the foreign currency.

When computing your delta it is important to know what currency was used to pay for the currency option. Returning to the stock analogy, suppose you paid for an IBM call option in IBM stock that you borrowed in the *stock-borrow* market. Then I would inherit a long delta position from the option and a short delta position position from the option payment. My overall net delta position will still be long (why?), but less long that it would have been if I had paid for it in dollars. The same is true if you pay for a currency option in units of foreign currency. When an option premium is paid in units of the foreign currency and the delta is adjusted to reflect this, we sometimes refer to it as the **premium-adjusted delta**.

As is probably clear by now, currencies can be quite confusing! And it takes time working in the FX markets before most people can get completely comfortable with the various market conventions. The one advantage that currencies have over stocks from a modeling perspective is that currencies do not pay discrete dividends and so the (often) ad-hoc methods that have been developed for handling discrete dividends are not needed when modeling currencies.

Other Deltas

There are other definitions of "delta" that are commonly used in FX-space. These alternative definitions arise because (i) the value of currency derivatives can be expressed naturally in domestic or foreign currency units and (ii) forwards play such an important role in FX markets. In addition to the regular Black-Scholes delta (which is the usual delta sensitivity) and the premium-adjusted delta we also have:

- The *forward* delta. This is the sensitivity of the option price with respect to changes in the value of the underlying forward contract with the same strike and maturity as the option.
- The *premium-adjusted* forward delta. This is the same as the forward delta but you need to adjust for the fact that the option premium was paid in units of the foreign currency.

Note that all of the delta definitions we have discussed thus far have implicitly assumed that we want to hedge the *domestic* value of the option. However, we might also want to hedge the *foreign* value of the option.

Example 1 (Hedging Domestic Value v Foreign Value)
Suppose an FX trader in Australia buys a JPY/AUD call option from an FX trader in Japan. If the Australian trader hedges the AUD value of the option and the JPY trader hedges the JPY value of the option, then they will *not* put on equal and opposite hedges. For example, suppose the value of the option is 100 AUD and the current AUD/JPY exchange rate is 65 JPY per AUD. If the Australian trader has hedged the AUD value of the option and the exchange rate moves from 65 to 67, say, his position will still be worth 100 AUD. However, the JPY value of his position will have changed from $100 \times 65 = 6,500$ JPY to $67 \times 100 = 6,700$ JPY. So hedging the AUD value of his position clearly does not hedge the JPY value of his position. ∎

As a result, we have another set of deltas that correspond to hedging the foreign value of the option. There are also "sticky-by-delta" versions of all these deltas. Clearly then, one must be very careful in specifying and interpreting a delta. The default delta that is quoted in the market place will, as always, depend on the currency pair and market conventions. Trading systems will generally be capable of reporting these different deltas.

What Does At-the-Money (ATM) Mean?

The usual definition of "at-the-money" in equity-space is that the strike of the option under consideration is equal to the current spot price. But there are other alternative definitions that commonly occur in FX-space. They are:

1. *At-the-money spot* which is our usual definition.
2. *At-the-money forward.*
3. *At-the-money value neutral*. This is the strike, K, such that the call value with strike K equals the put value with the same strike, K.
4. *At-the-money delta-neutral*. This is the strike, K, where the delta of the call option with strike K is equal to minus the delta of the put option with the same strike K. But as there many different definitions of delta, we have many possibilities here.

Building an FX Volatility Surface

The construction of a volatility surface in FX space is unfortunately much more difficult than in equity-space. The simple reason for this is that the liquid options in the FX markets are quoted for a given delta and not for a given strike. In particular, FX markets typically quote *volatility* prices for

1. ATM options. However, the particular definition of ATM will depend on the currency-pair and market conventions.
2. 25-delta and 10-delta strangles. A strangle is a long call option together with a long put option.
3. And 25-delta and 10-delta risk-reversals. A risk reversal is a long call option together with a short put option.

In each of the strangles and risk reversals, the strike of the put option will be lower than the strike of the call option. These strikes then need to be determined numerically using combinations of the quoted volatilities and the designated deltas. Once the various strikes and their implied volatilities have been calculated, curve fitting techniques are then used to build the entire volatility surface. So building an FX implied volatility surface is considerably more complicated than building an equity implied volatility surface. This appears to be a legacy of how FX options markets developed and in particular, of how risk reversals and strangles became the liquid securities in the market.

Until recently there have been very few (if any) detailed descriptions of the various conventions in FX markets and how FX volatility surfaces are built. More recently, however, some work[7] has been published on this topic. They can be consulted for further (tedious but necessary) details.

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2 The Triangle Relationships

The fact that all currency pair combinations can be traded imposes restrictions on the dynamics of the various exchange rates. To see this we will introduce some new notation. In particular, let

$$X_t^{a/b} \; := \; \#\text{ of units of currency A required to purchase } 1 \text{ unit of currency B.}$$

Then it is easy to see that

$$X_t^{a/b} = X_t^{a/c} \times X_t^{c/b} \tag{4}$$

$$X_t^{a/b} = \frac{1}{X_t^{b/a}} \tag{5}$$

The advantage of this notation is that it follows the usual rules of division, i.e. $a/c \times c/b = a/b$. The identity in (4) clearly defines the price process for any currency pair once the price processes for the other two pairs are specified.

Example 2 (Geometric Brownian Motions)
Suppose $X_t^{a/c}$ and $X_t^{c/b}$ both follow GBM's under some probability measure, $\mathcal{P}$, so that

$$\begin{aligned} dX_t^{a/c} &= \mu^{a/c} X_t^{a/c}\, dt + \sigma^{a/c} X_t^{a/c}\, dW_t^{a/c} \\ dX_t^{c/b} &= \mu^{c/b} X_t^{c/b}\, dt + \sigma^{c/b} X_t^{c/b}\, dW_t^{c/b} \end{aligned}$$

where $W_t^{a/c}$ and $W_t^{c/b}$ are $\mathcal{P}$-Brownian motions with $dW_t^{a/c} \times dW_t^{c/b} = \rho\, dt$. Then Itô's Lemma applied to (4) implies

$$\begin{aligned} dX_t^{a/b} &= \left(\mu^{a/c} + \mu^{c/b} + \sigma^{a/c}\sigma^{c/b}\rho\right) X_t^{a/b}\, dt \;+\; \sigma^{a/b} X_t^{a/b}\, dW_t^{a/b} \\ \text{where} \quad \sigma^{a/b} &:= \sqrt{(\sigma^{a/c})^2 + (\sigma^{c/b})^2 + 2\rho\sigma^{a/c}\sigma^{c/b}} \end{aligned} \tag{6}$$

and where $W_t^{a/b}$ is also[8] a $\mathcal{P}$-Brownian motion. The volatility, $\sigma^{a/b}$, is therefore completely determined by the volatilities and correlation of the other two exchange rates.

In practice, we of course know that FX rates do not follow GBM's. In particular, the implied volatilities for each of the three currency pairs will be a function of strike and time-to-maturity and so (6) is of little practical value. However, we emphasize again that if we have a good model for the dynamics of $X_t^{a/c}$ and $X_t^{c/b}$ then (4) uniquely determines the dynamics of $X_t^{a/b}$. ∎

Converting to Base Currency Deltas

Suppose now that you have bought a call option on $X_T^{a/b}$ and that the option payoff is denominated in the base domestic currency, A. Let $C(X_t^{a/b})$ denote the time t value (in currency A) of this option as a function of the

[7] See, for example, "FX Volatility Smile Construction" by D. Reiswich and U. Wystup (2009). See also "Foreign Exchange Option Pricing: A Practitioner's Guide" by I.J. Clark and published by Wiley.

[8] See (13) and the paragraph following it for a justification of this statement.

current exchange rate, $X_t^{a/b}$. Then this option could be delta-hedged in the usual Black-Scholes manner. Note, however, that if we delta-hedge the option we are hedging the *currency A* value of the option. In certain circumstances, however, and as mentioned earlier, we may wish to hedge or monitor the *currency C* value of this option. For example, a trading desk of a US bank may be responsible for all EUR/JPY trading. The trading book and its P&L will then be denominated in EUR, say[9]. The desk, however, may also want to monitor the USD value of its positions and P&L if USD is the base currency of the bank. After all, there is not much point in being the greatest EUR/JPY trader in the world and earning billions of JPY if the value of JPY against the USD has collapsed and is now worthless.

So let $V_t^c\left(X_t^{c/a}, X_t^{c/b}\right)$ denote the time t currency C value of the option. Then, using (4) and (5) we obtain

$$
\begin{aligned}
V_t^c\left(X_t^{c/a}, X_t^{c/b}\right) &= X_t^{c/a} \times C\left(t, X_t^{a/b}\right) \\
&= X_t^{c/a} \times C\left(t, X_t^{a/c} X_t^{c/b}\right) \\
&= \frac{C\left(t, X_t^{a/c} X_t^{c/b}\right)}{X_t^{a/c}} \qquad (7) \\
&= X_t^{c/a} \times C\left(t, \frac{X_t^{c/b}}{X_t^{c/a}}\right). \qquad (8)
\end{aligned}
$$

According to (8), or (7) if we prefer, the currency C value of the option now has two delta exposures: **(i)** a delta to the $X_t^{c/a}$ exchange rate and **(ii)** a delta to the $X_t^{c/b}$ exchange rate. We obtain

$$
\frac{\partial V_t^c}{\partial X_t^{c/a}} = C - \frac{X_t^{c/b}}{X_t^{c/a}} \frac{\partial C}{\partial X} \qquad (9)
$$

$$
\frac{\partial V_t^c}{\partial X_t^{c/b}} = \frac{\partial C}{\partial X} \qquad (10)
$$

where $\frac{\partial C}{\partial X}$ is the usual Black-Scholes delta. These quantities can then be used to hedge the currency C value of the option. Or we might just hedge the $X_t^{c/a}$ exposure and leave the $X_t^{c/b}$ exposure un-hedged. This way of viewing the value of the option is particularly useful if the trading book contains positions in many currency pairs and we ultimately only care about the value of the portfolio or the P&L in one particular currency. It is also useful when some of the currency pairs, e.g. $X_t^{a/b}$, are not liquid and we can only hedge them by trading in $X_t^{c/a}$ and $X_t^{c/b}$.

Remark 1 *If we are assuming sticky-by-delta implied volatility surfaces then we might want to include additional terms in (9) and (10) corresponding to $\frac{\partial C}{\partial \sigma^{a/b}}$. For example, if $X_t^{c/a}$ changes (and we hold constant $X_t^{c/b}$) then $X_t^{a/b}$ will change accordingly with a corresponding change in $\sigma^{a/b}$ due to the sticky-by-delta assumption. This would require an additional term on the right-hand-side of (9). Similarly, a change in $X_t^{c/b}$ whilst holding $X_t^{c/a}$ constant would require an additional term on the right-hand-side of (10). Whether or not such terms would be included in practice probably depends on the currencies in question, the preferences of the trader and the flexibility of the systems.*

[9] Although it could just as easily be JPY.

3 Options on Foreign Assets Struck in Domestic Currency

Let S_t denote the time t price of the foreign asset in units of the foreign currency. Then the payoff of the option is given by

$$\text{Payoff} \;=\; \max(0,\; X_T S_T - K). \tag{11}$$

Let us write the **domestic** risk-neutral dynamics for S_t, with the usual domestic cash account as numeraire, as

$$dS_t \;=\; \mu_s S_t\, dt + \sigma S_t\, dW_t \tag{12}$$

with $dW_t \times dW_t^{(x)} = \rho_t\, dt$. We assume the exchange rate dynamics continue to follow (2). In general we could let the correlation coefficient, ρ_t, be stochastic but we will assume here that it is a constant, ρ, for all t. Note that the underlying price, $X_t S_t$ is the time t value of the foreign stock measured in units of the domestic currency. We will also assume that S_t pays a continuous dividend yield of q.

Remark 2 *From the perspective of a foreign investor with foreign cash account as numeraire, the foreign asset's risk-neutral drift in (12) would be $\mu_s = r_f - q$. This is not true for the domestic investor, however, since S_t is not the (domestic) price of a traded asset nor*[10] *is the foreign cash account a domestic numeraire. In fact it is $S_t X_t$ that is the (domestic) price of a traded asset. This traded asset pays a dividend yield of q and so it is this security that has a drift of $r_d - q$ under the domestic risk-neutral dynamics, i.e. the $\mathcal{Q}$-dynamics taking the domestic cash account as numeraire .*

A simple application of Itô's Lemma implies that $Z_t := X_t S_t$ has dynamics given by

$$\begin{aligned}\frac{dZ_t}{Z_t} &= (r_d - r_f + \mu_s + \rho\sigma_x\sigma)\, dt + \sigma_x\, dW_t^{(x)} + \sigma\, dW_t \\ &= (r_d - r_f + \mu_s + \rho\sigma_x\sigma)\, dt + \sigma_z\, \left(\frac{\sigma_x}{\sigma_z}\, dW_t^{(x)} + \frac{\sigma}{\sigma_z}\, dW_t\right)\end{aligned} \tag{13}$$

where $\sigma_z := \sqrt{\sigma_x^2 + \sigma^2 + 2\rho\sigma_x\sigma}$. The quadratic variation of the second term in parentheses on the right-hand-side of (13) equals dt. We can therefore apply Levy's Theorem and write Z_t as a $\mathcal{Q}$-GBM with drift $r_d - q$ and volatility, σ_z. Note that we didn't care about the drift in (13) since the $\mathcal{Q}$-drift of Z_t must be $r_d - q$. The option price is then given by the Black-Scholes formula with risk-free rate r_d, dividend yield q and volatility σ_z. It is straightforward (why?) to see that the value of μ_s in (13) satisfies

$$\mu_s \;=\; r_f - q - \rho\sigma_x\sigma. \tag{14}$$

Remark 3 *In practice, a more sophisticated model would generally be used to price this option, depending as it does on the joint dynamics of X_t and S_t. This is not true of our first example where we priced a vanilla currency option. In this case the correct volatility to be used in the Black-Scholes formula could be read off the volatility surface. It is also not true of the option in the exercise below where the volatility surface can again be used to obtain the correct option price.*

Remark 4 *Note that in the U.S a security whose time t value is given by $X_t S_t$ is called an* American Depository Receipt *or* ADR. *If, for example, S_t represents the time t value of a share of Volkswagen denominated in Euro, then a Volkswagen ADR allows US investors to easily invest in Volkswagen without having to trade in the foreign exchange markets. Similarly a* European Depository Receipt *or* EDR *allows European investors to invest directly in non-Euro-denominated securities. Note, however, that the holder of an ADR or EDR is still exposed to foreign exchange risk. This is in contrast to the holder of a* quanto *security which we will consider in Section 4.*

[10] In particular, an EMM-numeraire pair for a foreign investor is **not** an EMM-numeraire for a domestic investor. But see Exercise 5.

4 Quanto-Securities

We now consider securities whose payoff is based on S_T converted into units of domestic currency at a **fixed** exchange rate, $\bar{X}$. Such a security is called[11] a *quanto*. We will consider quanto forwards and quanto options and price them within the Black-Scholes framework. Quantos are traded very frequently in practice, particularly in the structured products market. Consider, for example, an option on a basket of three stocks: IBM, Toyota and Siemens. The three stock prices are denominated in USD, JPY and EUR, respectively. But what currency should the option payoff be denominated in? Suppose we choose US dollars but we don't want any exchange rate exposure. Then a call option with strike K might have a time T payoff given by

$$\text{Payoff}_T \ := \ \max\left(0,\ c_0\text{IBM}_0 + c_1\text{SIE}_T + c_2\text{TOY}_T \ - \ K\right) \tag{15}$$

and we say that Siemens and Toyota have been *quanto'ed* into US dollars. More specifically, we can also rewrite the payoff in (15) as

$$\text{Payoff}_T \ := \ c_0 \ \max\left(0,\ \text{IBM}_0 + \frac{c_1}{c_0}\text{SIE}_T + \frac{c_2}{c_0}\text{TOY}_T \ - \ \frac{1}{c_0}K\right) \tag{16}$$

which can now be interpreted as c_0 call options on a basket of IBM, Siemens and Toyota with a strike of K/c_0 and where Siemens and Toyota have been *quanto'ed* into US dollars at fixed exchange rates of c_1/c_0 and c_2/c_0, respectively.

Another market where quantos are frequently traded is the commodity market. Most commodities are priced in US dollars but non-US investors often wish to trade commodities denominated in units of their domestic currency. For example, a European investor who wishes to buy an option on crude oil but without the USD/EUR exchange risk could buy a quanto option where the oil is quanto'ed into EUR. For example, a call option on oil with strike K and maturity T would normally have a time T payoff of $\max(0,\ O_T - K)$ which is denominated in USD since the oil price, O_T, is denominated in USD. If the option is quanto'ed into Euro, however, then the payoff is again $\max(0,\ O_T - K)$, but now it is denominated in Euro despite the fact that O_T is denominated in USD.

Quanto Forwards

Consider a quanto-forward contract where $\bar{X}S_T$ is exchanged at maturity, T, for F^q units of the domestic currency. If we enter into this contract at time t, then we know that $F_t^q := F^q$ is chosen so that the initial value of the contract is 0. Using the usual domestic cash account as numeraire, we immediately obtain that

$$0 = \mathbf{E}_t^{\mathcal{Q}}\left[e^{-r_d(T-t)}(\bar{X}S_T - F_t^q)\right]$$

so that $F_t^q = \bar{X}\mathbf{E}_t^{\mathcal{Q}}\left[S_T\right]$. We therefore need the $\mathcal{Q}$-dynamics of S_t in order to compute F_t^q. But from (12) and (14), we know the $\mathcal{Q}$-dynamics of S_t are

$$dS_t \ = \ (r_f - q - \rho\sigma_x\sigma)S_t\ dt + \sigma S_t\ dW_t \tag{17}$$

and so we obtain (why?)

$$F_t^q = \bar{X}S_t e^{(r_f - q - \rho\sigma_x\sigma)(T-t)}. \tag{18}$$

It is also straightforward (why?) to show that the fair time-t value of receiving $\bar{X}S_T$ at time T is given by

$$V_t = \bar{X}S_t e^{(r_f - r_d - q - \rho\sigma_x\sigma)(T-t)}. \tag{19}$$

[11] Our treatment of quanto securities draws from "A Course in Derivative Securities" by Kerry Back.

Note that V_t is the value in units of the domestic currency of a non-dividend[12] paying traded asset. Note also that

$$F_t^q = e^{r_d(T-t)} V_t \tag{20}$$

which is the usual form of a forward price. That is, F_t^q is the time-t forward price for time-T delivery of a domestic non-dividend paying security that is worth V_T at time T. The standard replicating portfolio argument using V_t and the domestic cash account can be used as an alternative proof of (20).

The question also arises as to how to dynamically replicate the time-T payoff $V_T = \bar{X} S_T$ using the domestic cash account, the foreign cash account and the foreign asset. In fact at each time, t, the replicating strategy consists of

1. Investing V_t units of the domestic currency in the foreign asset.
2. Investing $-V_t$ units of the domestic currency in the foreign-cash account, i.e. borrowing in the foreign currency to fund the purchase of the foreign asset in step 1.
3. Investing V_t units of the domestic currency in the domestic cash account.

Note that the first two components of the hedging strategy ensure that the strategy has no FX exposure. This must be the case as the asset we seek to replicate, $V_T = \bar{X} S_T$, has no FX exposure. Note also that the time t-value of the replicating portfolio is V_t, as expected.

Quanto Options

We are now in a position to price quanto options. In fact, given our earlier analysis pricing a quanto option is straightforward. Let us assume in particular that we seek to price a call option with payoff in units of the domestic currency given by

$$\text{Quanto Option Payoff} \;=\; \max\left(0,\ \bar{X} S_T - K\right).$$

Martingale pricing states that the time-t price, P_t say, of this security is given by

$$P_t \;=\; \mathbf{E}_t^{\mathcal{Q}}\left[e^{-r_d(T-t)} \max\left(0,\ \bar{X} S_T - K\right)\right]$$

where the $\mathcal{Q}$-dynamics of S_t satisfy (17). Note that we can rewrite (17) as

$$dS_t \;=\; (r_d - q_f) S_t\, dt + \sigma S_t\, dW_t \tag{21}$$

where $q_f := q + r_d - r_f + \rho \sigma_x \sigma$. But then we see that the quanto option price is just $\bar{X}$ times the Black-Scholes price of a call option on a stock with initial value S_t, time-to-maturity $T - t$, volatility σ, strike $K/\bar{X}$ and dividend yield q_f.

Remark 5 *An alternative method of calculating the quanto option price is to view it as an option on the non-dividend security with time t-value V_t. We can again use the Black-Scholes formula but this time using a zero-dividend yield and an initial value of $V_0 = \bar{X} S_0 e^{(r_f - r_d - q - \rho\sigma_x\sigma)T}$.*

It is also straightforward to replicate the payoff of the quanto option using the foreign asset, the foreign cash account and the domestic account. This is done by adopting the usual Black-Scholes delta-hedging strategy using the domestic cash account and the portfolio with value V_t as the underlying risky asset. At each time t, we hold delta units of the portfolio and borrow the difference between the cost of the delta units and the value of the option. The portfolio is constructed using steps 1 to 3 as described earlier.

Remark 6 *Note that the key to valuing quanto'ed securities is obtaining the domestic risk-neutral dynamics of these securities. Everything else follows from the usual arguments.*

Remark 7 *Banks and other market makers are very reluctant to quanto assets from "high-risk" economies. After all, who would want to sell an option that pays in US dollars $\max(0,\ Z_T - K)$ where Z_T is the return on the Zimbabwe stock market!?*

[12] Note that even though V_t is non-dividend paying, the underlying security S_t, may well pay dividends.

Exercises

1. Prove (1) directly using a no-arbitrage argument.

2. Compute the time t value of a forward contract that was initiated at time $t = 0$ with strike K for delivery at time $T > t$ of a foreign currency.

3. Create an Excel spreadsheet to price and hedge currency options. You can assume the options are on the JPY/EUR exchange rate. Simulate values of the exchange rate through time and execute the usual delta-hedging strategy. Compare the final USD value of the hedge portfolio (which is denominated in JPY) with the final USD value of the portfolio where the USD-value of the option was dynamically hedged. You can assume the exchange rates all follow GBM's with constant interest rates, volatilities and correlation. These parameters should be inputs in the spreadsheet. (*Hint*: Equations (7) through (10) should be useful.)

4. Explain why the value of μ_s in (13) satisfies $\mu_s = r_f - q - \rho\sigma_x\sigma$.

5. Let $\mathcal{Q}^f$ be an EMM for a foreign investor corresponding to some foreign numeraire, X^f say. Note that X_t^f is then denominated in units of the foreign currency. Show that $\mathcal{Q}^f$ can also be used as an EMM for a *domestic* investor. What is the corresponding *domestic* numeraire? Is the converse also true, i.e., can any domestic EMM also be used as an EMM for a foreign investor?

6. How would you price an option on a foreign asset that has been struck in foreign currency?

7. In Section 3 we discussed the pricing of a call option on a foreign asset that was struck in units of the domestic currency, i.e. an ADR option. We showed that such an option could be priced using the Black-Scholes formula with suitably adjusted parameters. Describe how you would delta-hedge this option. In particular, what underlying securities would you need to trade and how much of each security would you hold at each time t for $0 \leq t \leq T$?

   Hint: An easy way to construct the self-financing strategy is to write the time t option price as $C(t, Z_t)$ where $Z_t := X_t S_t$. Since Z_t is a traded asset (it is the domestic value of holding the foreign asset) we can adopt the usual Black-Scholes hedging strategy with Z_t as our underlying security.

8. **(a)** Let X_t and Y_t be the time t prices of two non-dividend-paying assets (named asset one and asset two, respectively) with price dynamics under some probability measure, $\mathcal{P}$, given by

$$
\begin{aligned}
dX_t &= \mu_y X_t \, dt + \sigma_x X_t \, dW_t^{(x)} \\
dY_t &= \mu_y Y_t \, dt + \sigma_y Y_t \, dW_t^{(y)}
\end{aligned}
$$

   where $W_t^{(x)}$ and $W_t^{(y)}$ are (possibly correlated) $\mathcal{P}$-Brownian motions. Suppose now that your portfolio at time t holds $\phi_t^{(x)}$ units of the first asset, $\phi_t^{(y)}$ units of the second asset and η_t units of the domestic cash account. Let P_t be the time t value of this portfolio. If the portfolio is self-financing, show that

$$
dP_t = P_t \left[r + \theta_t^{(x)}(\mu_x - r) + \theta_t^{(y)}(\mu_y - r) \right] dt + \theta_t^{(x)} \sigma_x P_t \, dW_t^{(x)} + \theta_t^{(y)} \sigma_y P_t \, dW_t^{(y)} \qquad (22)
$$

   where r_t is the domestic risk-free interest rate and $\theta_t^{(x)}$ and $\theta_t^{(y)}$ are the time-t fractions of wealth invested in the first and second assets, respectively.

   (b) Let $C(t, X_t, Y_t)$ be the time t price of a derivative security that expires at time $T > t$ with payoff $C(T, X_T, Y_T)$ and that does not pay intermediate cash-flows between times t and T. Use Itô's Lemma and (22) to show that at each time t the self-financing portfolio that replicates the derivative security holds $\partial C/\partial X_t$ units of the first asset and $\partial C/\partial Y_t$ units of the second asset. (Hint: This only takes two lines but relies on the fact that if you have two diffusions, C_t and P_t say, that are identical then their dt, dW_t etc coefficients are also identical.)

Remark 1: Note that if the securities had dividend yields then we could simply account for them in (22) via the self-financing condition for dividend paying securities. (In Exercise 9 below two foreign exchange rates will play the roles of X_t and Y_t. In that case the securities pay dividends according to the two risk-free interest rates. These dividends need to be included in the self-financing strategy that you simulate in Exercise 9.)

Remark 2: Suppose the value of the derivative security depended on another source of noise, Z_t say, but that no financial asset depended on Z_t. Then we would not have been able to replicate the derivative security and so the market would be incomplete in that case. This follows since the time t price of the derivative security would now be $C(t, X_t, Y_t, Z_t)$ and we would have a dZ_t term in the dynamics for C_t that could not be matched by a corresponding term in (22).

9. **(a)** Suppose you are an FX options trader with responsibility for options on the JPY/EUR exchange rate. Write a program that simulates a delta-hedging strategy where you hedge (i) the EUR value of the option and (ii) the USD value of the option, adopting a different strategy in each case. Note that in case (ii) you will need to trade USD/EUR **and** USD/JPY at each time t. Compare the final values of each of the two hedging strategies in each of the two currencies. You can assume that options are priced assuming the exchange rates all follow GBM's with constant interest rates, volatilities and correlation. These parameters, as well as the option expiration, T, option strike, K, and number of re-balancing periods, M, should be inputs to your code.

   Hint: Let Currency *a* refer to EUR, currency *b* refer to JPY and currency *c* refer to USD. Let ρ, $\sigma^{a/c}$ and $\sigma^{c/b}$ be inputs to your code. Then $\sigma^{a/b}$ can be calculated from equation (6) in the lecture notes. Other inputs will include time-to-maturity T, option strike K, the risk-free interest rates in each currency as well as the initial exchange rates. Let us assume that we are hedging a call option with a time t EUR price of $C(t, X_t^{a/b})$. (Note that if we had set *a* to denote JPY and *b* to denote EUR then the price $C(t, X_t^{a/b})$ would be denominated in JPY and so we would have needed to multiply it by the appropriate exchange rate to convert it into EUR. It is fine to do it this way but clearly not as straightforward.)

   Note that Exercise 8 can be used together with equations (7) through (10) to do part (ii). Remember that your delta-hedging strategies should be self-financing. You should also assume that the initial value of the hedging portfolio is equal to the initial EUR value of the option in case (i) and the initial USD value of the option in case (ii). Of course the initial USD value is simply the initial EUR value times the initial USD/EUR exchange rate.

   Finally, it is worth mentioning that you can assume any processes you like for *simulating* the hedging strategy as long as you respect the self-financing condition. You will correctly replicate the payoff of the option (as $M \to \infty$) *only* if you assume GBM's with the correct volatilities and correlation coefficient.)

   (b) Adjust your code (by simply adding a "for loop" and a few additional lines) so that you can simulate N paths of your hedging strategies and compute the average tracking error in each of the two cases. The average tracking error is the average of the absolute difference between the value of the hedging strategy and the payoff of the option. N should also be an input to your code. (You can check your code is correct by confirming that the tracking error goes to zero as $N, M \to \infty$.)

10. When you are hedging the USD value of the option in Exercise 9 (a)(i), which of the parameters $\sigma^{a/b}$, $\sigma^{a/c}$, $\sigma^{c/b}$ and ρ is it essential for the hedger to estimate correctly in order to ensure there is no

replication error in the limit as $M \to \infty$? Use your code from (b) to check your answer.

11. Referring to Section 4, show that the fair time-t value of receiving $\bar{X}S_T$ at time T is given by

$$V_t = \bar{X}S_t e^{(r_f - r_d - q - \rho\sigma_x\sigma)(T-t)}. \tag{23}$$

Hint: This requires a one-line argument given (17) or (18).

12. Use Itô's Lemma and (19) to show that the replicating strategy for quanto forwards is as described in Section 4.

Hint: You might start by assuming that a_t units of the domestic currency are invested in the foreign asset, b_t units of domestic currency are invested in the foreign cash account and c_t units of domestic currency are invested in the domestic cash account. The time-t domestic currency value of this portfolio is $W_t = a_t + b_t + c_t$. The replicating strategy must be self-financing and so the dynamics of the strategy must satisfy

$$dW_t \;=\; a_t\ d(\text{Return on foreign asset})_t + b_t\ d(\text{Return on foreign cash account})_t + c_t\ d(\text{Return on domestic cash account})_t.$$

Note, for example, that $d(\text{Return on foreign asset})_t = d(X_t e^{qt} S_t)/(X_t e^{qt} S_t)$. Why? Use this and other corresponding terms to compute dW_t and then compare with the dynamics of V_t which are obtained using (19). (See Section 6.6 of "A Course in Derivative Securities" by Kerry Back if you are unsure.)

13. Write a program to simulate the delta-hedging strategy of a quanto-call option. You can assume the exchange rate and foreign asset value all follow GBM's with constant interest rates, volatilities and correlation. As in Exercise 9, your code should be able to simulate N paths and you can check that your code is correct by checking that the tracking error goes to zero as $N, M \to \infty$.

Hint: Take V_t as the underlying non-dividend paying security for the quanto-call option as mentioned in Remark 5 of the lecture notes. Then, equations (12) and (14) together with Itô's Lemma applied to equation (19) imply (as expected since V_t is a traded asset) that

$$dV_t = r_d V_t\ dt \ + \ \sigma V_t\ dW_t.$$

So we can delta hedge the option using V_t as the underlying in the usual Black-Scholes manner. At time t suppose we need to hold ϕ_t units of V_t. Then we achieve this by holding ϕ_t units of the quanto forward. But Exercise 12 tells us how to replicate the forward using the foreign currency, the foreign asset and domestic cash account and so we know how to replicate the quanto-option.