IEOR E4706: FE: Discrete-Time Asset Pricing (Fall 2004)

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Assignment 4: Due Friday October 22th

The questions below refer to the material of the *Martingale Pricing Theory* lecture notes. It will save time if you use software such as Matlab for solving the systems of linear equations that might arise.

Question 1
The single-period model of Example 7 is a complete market. Find the replicating portfolio for each of the elementary securities.

Question 2
Do Exercise 5. That is, show that if a single-period model is incomplete then at least one elementary security is not attainable.

Question 3
(a) Referring to Example 7, find a set of risk-neutral probabilities for the case where we take the 2^{nd} security as numeraire. (Recall that the cash account is the 0^{th} security so the 2^{nd} security is the security with price 2.4917 at date $t = 0$.)

(b) Are these risk-neutral probabilities unique? Explain your answer.

(c) Would we get the same set of risk-neutral probabilities if we used a different numeraire?

Question 4
Suppose we find ourselves at node $I_1^{6,7,8,9}$ at $t = 1$ in Example 10. The one-period model beginning at this node is incomplete and so not every security is attainable. Is a call option on the second security with strike $k = 2$ attainable? If so, what is the price of this call option? If not, can you bound the price of the option, i.e. find bounds on the option price that must be satisfied if there is to be no arbitrage?

Question 5
Do Exercise 3. That is, show that if we did not insist on each q_k being strictly positive in Definition 8 then Proposition 1 would not hold.

Question 6
Referring to Example 9, compute the date $t = 0$ value of a European style option that expires at $t = 2$ with a payoff function given by $\max\left(0,\ S_2^{(1)} - S_2^{(2)} - 1\right)$.

Question 7
Find the replicating portfolio for the contingent claim $X = \max\left(0,\ S_2^{(1)} - S_2^{(2)} - 1\right)$, i.e. the option of the previous question.

Question 8
Consider an equity swap based on the first security of Example 9. (Recall that the securities are numbered 0, 1 and 2 so that the first security refers to the security whose value at node $I_1^{1,2,3}$, for example, is 1.4346.) Party A is long the swap and receives $\$100 \times S_t^{(1)}$ from party B at dates $t = 1$ and $t = 2$. In return, party B is short the swap and receives $\$M$ at dates $t = 1$ and $t = 2$ from party A. Compute the value of the swap at $t = 0$ and determine what value of M will make the swap worth 0 at $t = 0$. (Note that you may answer this question using either the state prices or the unique EMM, Q, that corresponds to your choice of numeraire security. Note also that we do not need to use any knowledge of forward prices to compute the value of the swap.)

Question 9
Referring to Example 10, suppose we changed the payoffs of the 1^{st} and 2^{nd} securities in states ω_4 and ω_5 so that the model was arbitrage-free. (By leaving the payoff of the 0^{th} security unchanged at 1.1025 in those states, we maintain its interpretation as the cash account.)

(a) Prove that a zero-coupon bond with face value F and maturity $t = 2$ is attainable.

(b) Now prove that the payoff of a forward contract, with any one of the three securities as the underlying, is attainable. In particular, conclude that we can compute unique arbitrage-free forward prices.

Question 10
We have seen that the price, P_0, of an attainable security (with date $t = 1$ payoff P_T) in an arbitrage-free model is given by

$$\frac{P_0}{S_0^{(n)}} = \mathrm{E}_0^Q\left[\frac{P_T(\omega)}{S_T^{(n)}(\omega)}\right]$$

where $S_t^{(n)}$ is the date t price of the numeraire security, Q its corresponding EMM and where we have assumed that the security has no intermediate cash-flows in $[0, T]$. Show that we can also express the security price as

$$P_0 = \mathrm{E}_0^P\left[\frac{q(\omega) S_0^{(n)}}{p(\omega) S_T^{(n)}(\omega)} P_T(\omega)\right].$$

(Note that $qS_0^{(n)}/pS_T^{(n)}$ is a *stochastic discount factor*, generalizing the usual notion of a deterministic discount factor.)